Electricity and Magnetism

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Contents

1 Electric Charge and Electric Field

Lecture 1

2026-01-06

Charge is the fundamental property of particles like protons ($+e$) and electrons ($-e$) to experience a force in an electric field. Early demonstrations of rubbing fur against a plastic rod and watching the attraction/repulsion of the newly charged objects describes the transfer of charge up the triboelectric series. In such an experiment, charge is conserved.

In 1909, Millikan demonstrated that all charges are an integer multiple of the fundamental charge of the electron (Thomson discovered the electron in 1897), which is $e = 1.6 \cdot 10^{-19} C$. The Coulomb C is the unit of electric charge. This charge can move freely through conductors, but not through insulators.

From this, we can complete *charging by induction*, in which a charged object near a conductor will induce a separation of charge. If the conductor is then grounded, some of the accumulated separated charges will flow into the ground, leaving a *net charge* on the conductor. Similarly, bringing a charged object near an insulator will *polarize* the insulator, resulting in a net force.

Definition 1. Coulomb's Law states that the magnitude of force between two point charges is given by:

$$F = k\frac{q_1 q_2}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2} \tag{1}$$

Where $r(m)$ is the separation of the charges, and $q_i(C)$ is the magnitude of the charges, and $\epsilon_0 = 8.854 \cdot 10^{-12}\frac{C^2}{Nm^2}$ is the vacuum permittivity. Opposite charges will attract (indicated by a negative magnitude).

Note that the electric force is strong!

Example. Take two He-4 nuclei, stripped of electrons. They have 2 protons and 2 neutrons for a net charge of $2e$, and each have a mass of $m \approx 4 \cdot 1.67 \cdot 10^{-27} = 6.68 \cdot 10^{-27} kg$. We can find the electric force between them as:

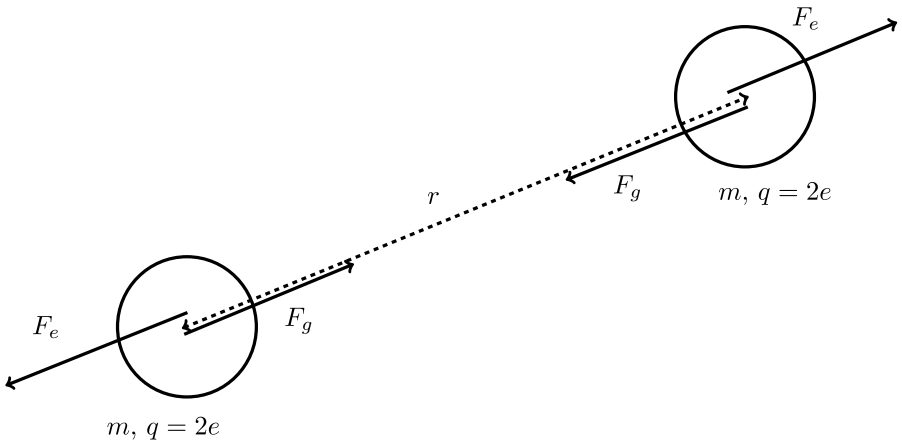

Figure 1: Alpha particle force comparison

$$F_e = \frac{1}{4\pi \cdot 8.854 \cdot 10^{-12}}\frac{\left(2 \cdot 1.6 \cdot 10^{-19}\right)^2}{r^2}.$$

The gravitational force between them is:

$$F_g = \frac{Gm^2}{r^2} = \frac{6.67 \cdot 10^{-11} \cdot 6.68 \cdot 10^{-27}}{r^2}.$$

Taking the ratio of these two forces, we get:

$$\frac{F_e}{F_g} \approx 3 \cdot 10^{35}.$$

$\lozenge$

Lecture 2

2026-01-08

From Coulomb's law, we know that the electric force is central and can be superimposed as:

$$\boldsymbol{F}_{net} = \sum_{i \neq j} \boldsymbol{F}_{ij}.$$

While a single charge exerts no self-force, it creates an *electric field* around it.

Definition 2. The electric field is a field surrounding a charged particle, with dimensions of force per unit charge, with direction defined by the direction a positive test charge would be pushed.

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \qquad \boldsymbol{F} = q_0\boldsymbol{E}.$$

1.1 Continuous charge distributions

The net electric field created by multiple charges is the superposition of the effects of each individual charge.

A uniform field has the same magnitude and direction everywhere in a region. The electric field at every point inside a conductor must be 0, as any non-uniformities would result in the movement of charges to equalize the potential. If we added a cavity into the conductor, the charge on this inner surface is also 0 *unless* there is a charge inside the cavity. In this case, an equivalent and opposite charge will manifest on the inner surface of the conductor This induced separation results in an equal charge adding to the outer surface of the conductor.

For charged objects with physical geometry, we use the concept of charge density to calculate the net field created by the object:

- Linear charge density λ is the charge per unit length, such as those along a thin rod;
- Surface charge density σ is the charge per unit area, such as that on a charged conductor;
- Volume charge density ρ is the charge per unit volume.

Lecture 3

2026-01-13

To find the net field created by a distributed charge, the principle of superposition becomes an integral over $dq = \rho dx$ for some density ρ per unit x a distance l from the considered location:

$$E = \frac{1}{4\pi\epsilon_0}\int_P \frac{\lambda\left(l\right)}{l^2}\,dl.$$

Generalizable from the linear case shown here to surfaces and volumes.

Example. Find the electric field at a point $(0, 0, z)$ above a rod of length L situated on the x-axis, with one end on the origin.

The uniform linear charge density λ yields $dq = \lambda dx$, as usual. The geometry of the rod-point system means that each dq is a distance $\sqrt{z^2 + x^2}$ from the point, where x is the x-coordinate of dq.

First, find the z-component of the net field:

$$\begin{aligned} E_z &= \frac{1}{4\pi\epsilon_0}\int_0^L \frac{\lambda}{z^2+x^2}\cdot\frac{z}{\sqrt{z^2+x^2}}\,dx \\ &= \frac{\lambda z}{4\pi\epsilon_0}\int_0^{\arctan\frac{L}{z}} \frac{z\sec^2\theta}{z^3\sec^3\theta}\,d\theta \\ &= \frac{\lambda}{4z\pi\epsilon_0}\sin\left(\arctan\frac{L}{z}\right) \\ &= \frac{\lambda}{4z\pi\epsilon_0}\frac{L}{\sqrt{z^2+L^2}}. \end{aligned}$$

The integral was evaluated using the substitution $x = z\tan\theta \implies dx = z\sec^2\theta d\theta$

Now find the x-component of the net field:

$$\begin{aligned} E_x &= \frac{1}{4\pi\epsilon_0}\int_0^L \frac{\lambda}{z^2+x^2}\frac{-x}{\sqrt{z^2+x^2}}\,dx \\ &= -\frac{\lambda}{4\pi\epsilon_0 z^3}\int_0^{\arctan\frac{L}{z}} z^2\sin\theta\,d\theta \\ &= \frac{\lambda}{4\pi\epsilon_0 z}\left(\cos\left(\arctan\frac{L}{z}\right)-1\right) \\ &= \frac{\lambda}{4\pi\epsilon_0 z}\left(\frac{z}{\sqrt{z^2+L^2}}-1\right). \end{aligned}$$

Observe that if L is small such that $O\left(L^2\right)$ is negligible, the field created by our tiny bar collapses to that of a point charge. ♢

Example. Find the magnitude of the electric field at some r away form an infinite charged line with charge density λ.

First, magnitude of E at any r from the wire:

$$\begin{aligned} \boldsymbol{E} &= \int_{-\infty}^{\infty} \frac{1}{4\pi\epsilon_0}\frac{\lambda}{l^2+r^2}\cdot\frac{r}{\sqrt{l^2+r^2}}\,dl \\ &= \frac{\lambda r}{4\pi\epsilon}\int_{-\infty}^{\infty}\frac{1}{(l^2+r^2)^{\frac{3}{2}}}\,dl \\ &= \frac{\lambda r}{4\pi\epsilon_0}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{r^2\sec\theta}\,d\theta \\ &= \frac{\lambda}{4\pi\epsilon_0 r}\cdot 2 \\ &= \frac{\lambda}{2\pi\epsilon_0 r}. \end{aligned}$$

Integral evaluated using $l = r\tan\theta$, $dl = r\sec^2\theta d\theta$, and $\sqrt{l^2+r^2} = r\sec\theta$. ♢

1.2 Field lines

We can visually represent a field using *field lines*, which point in the direction of the field, and whose density indicates field strength. Note that field line can never cross, and only terminate at a negative charge or → ∞.

1.3 Dipoles

An electric dipole is a system of two equal and opposite charges $+q$ and $-q$ separated by a distance $\boldsymbol{d}$, with a vector pointing from the negative to positive charge. A dipole placed in a uniform electric field experiences no net force, since the charges are equal and opposite, but it may experience a net torque. If the dipole is *not* aligned with $\boldsymbol{E}$, there is a net torque.

$$\boldsymbol{\tau} = 2\frac{\boldsymbol{d}}{2} \times q\boldsymbol{E} = q\boldsymbol{d} \times \boldsymbol{E} = \boldsymbol{p} \times \boldsymbol{E}.$$

Where $\boldsymbol{p} = q\boldsymbol{d}$ is the *electric dipole moment.* A dipole pointing in the same direction as $\boldsymbol{E}$ is in stable equilibrium, while a dipole (or dipole moment) pointing opposite to $\boldsymbol{E}$ is in unstable equilibrium. Any displacement from this anti-alignment will result in the field acting to align the dipole.

The work done to change between two configurations is:

$$W = \int_{\phi_1}^{\phi_2} -\tau \, d\phi = |\boldsymbol{p}||\boldsymbol{E}| \left(\cos\left(\phi_2\right) - \cos\left(\phi_1\right)\right).$$

We can define a potential energy as the change in energy (work done) to be:

$$U = -\boldsymbol{p} \cdot \boldsymbol{E}.$$

This is minimal for an aligned dipole, and maximal for an anti-aligned dipole.

Lecture 4

2026-01-15

1.4 Gauss' Law

Definition 3. The flux through a surface S is a measure of the amount of perpendicular vector field lines passing through the surface.

$$\Phi_E = \iint_A \boldsymbol{E} \cdot d\boldsymbol{A}.$$

Definition 4. Gauss' Law: the flux through any closed surface is proportional to the charge enclosed.

$$\Phi_E = \int_S \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q_{enclosed}}{\epsilon_0}.$$

For any closed surface, the flux through the surface is a measure of the charge enclosed inside, as field lines produced by a positive charge inside the surface either propagate to infinity, or terminate on a negative charge also inside the surface. Note that for any closed surface with a point charge $+q$ enclosed, the flux through this surface will always be the same. This stems from how the magnitude of the field $\propto \frac{1}{r^2}$, while the surface area $\propto r^2$, which cancel.

Note. For any closed surface,

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q_{enclosed}}{\epsilon_0}.$$

△

We can take our integral form of Gauss' Law and manipulate it into a differential form. First, apply divergence theorem, noting that the enclosing surface S is closed, positively oriented, and piecewise smooth, and our field has continuous partials on the region.

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{E}\, dv = \frac{Q_{enclosed}}{\epsilon_0}.$$

Writing the charge in terms of the charge density, $Q_{enclosed} = \int_V \rho\, dv$, we can obtain the differential form:

$$\iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{E} dv = \iiint_V \frac{\rho}{\epsilon_0} dv.$$

Since this holds for any volume, then our differential form becomes:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}.$$

In the case where a point charge is enclosed, we make clever use of the Dirac Delta function to represent our charge density.

First, we describe the total charge as $Q = \iiint_V \rho(\boldsymbol{r}) dV$, and then describe the charge density using the Dirac Delta function:

$$Q = \iiint_V q\delta^3 \left(r - r_0\right) d^3 r.$$

Where we take $r_0 = 0$ as the location of the charge q, without loss of generality. This matches what we get if we directly take the divergence the electric field:

$$\boldsymbol{E}\left(\boldsymbol{r}\right) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{r}}{r^2} \rho\left(\boldsymbol{r}\right) dv.$$

Note that the divergence of the electric field created by a point charge is infinite at the point charge, and 0 everywhere else.

Example. Calculate the electric field outside a uniformly charged sphere of radius R and charge q.

Apply Gauss' Law, noting that $\boldsymbol{E}$ will be parallel to the radial area element $d\boldsymbol{A}$, and integrate over a sphere:

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{A} = |E| 4\pi r^2 = \frac{q}{\epsilon_0}.$$

Therefore at some distance $\boldsymbol{r}$ from the centre of the charged sphere, the electric field is $\boldsymbol{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$ Note that this result is equivalent to if the charged sphere was replaced with a point charge at its centre. ♢

Note that this application of Gauss' Law shows how it is super useful for spherical symmetry, cylindrical symmetry, and planar symmetry (basically symmetry).

Note. The electric field on the surface of a conductor is always perpendicular to the surface of the conductor. Using a Gaussian surface, we can show that this field is given by:

$$\boldsymbol{E} = \frac{\sigma}{\epsilon_0}.$$

Where σ is the local charge density. △

Lecture 5

2026-01-20

2 Electric Potential

Observe that for $\boldsymbol{E}$, the curl is obviously 0 since it is spherically symmetric:

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}.$$

We can also show this by taking a line integral from general point a to point b in spherical coordinates, and show that the final result (work per unit charge done by $\boldsymbol{E}$) is path independent. Therefore, apply Stokes' theorem for work around a closed loop $\implies \boldsymbol{\nabla} \times \boldsymbol{E} = 0$.

Since $\boldsymbol{E}$ is a vector field that is defined on a simple connected domain ($\mathbb{R}^2 \setminus \boldsymbol{O}$) with $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$, it is a conservative vector field.

Definition 5. The electric field can be represented by the curl of a scalar function, the **electric potential**, in units of $\frac{N \cdot m}{C} \equiv \frac{J}{C} \equiv V$.

$$V(\boldsymbol{r}) = -\int_{\boldsymbol{O}}^{\boldsymbol{r}} \boldsymbol{E} \cdot d\boldsymbol{l}.$$

Where $\boldsymbol{O}$ is defined to have a potential of $V(\boldsymbol{O}) = 0$.

Typically, we set the potential equal to 0 at infinity. An exception to this would be an infinite charge distribution, where the potential blows up $r \to \infty$, in which case we can choose a different point.

Lecture 6

2026-01-22

Our potential function satisfies:

$$V(\boldsymbol{b}) - V(\boldsymbol{a}) = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E}\, d\boldsymbol{l} = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\nabla} V\, d\boldsymbol{l} \implies E = -\boldsymbol{\nabla} V.$$

This is the differential form of our line integral form.

A surface or contour over which the potential is constant is called an *equipotential*. On an equipotential, the potential doesn't change, so no work is done. If positive work is done by the system, the potential decreases (think conservation of mechanical energy). As such, electric field lines run perpendicular to equipotential lines.

This fact alone implies that electric field lines must always be perpendicular to every point along a conducting surface since it is at equipotential. This can all be drawn from the relation $\boldsymbol{E} = -\boldsymbol{\nabla} V$ since $\boldsymbol{E} = \boldsymbol{0} \implies V = \mathit{const}$, and $\boldsymbol{E}$ points in the direction of decreasing potential.

Example. Inside a conductor, the field is 0. Since $\boldsymbol{E} = -\boldsymbol{\nabla} V$, it makes sense that the potential inside a conductor is constant. This defines an equipotential region. Since potential obeys superposition, if we added another charge outside of our conductor, the inside of the conductor would still be equipotential, but at a different value. ◊

We can find the potential created by a point charge by integrating the negative of the electric field from our reference point ∞ to some distance r away from the point charge.

$$V\left(\boldsymbol{r}\right) = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r^2}\,dr = \frac{1}{4\pi\epsilon_0}\frac{q}{r}.$$

For a distributed charge (shown for 1D case):

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{1}{r}\,dq, \quad dq = \rho dl.$$

Recall that this is like the work done per unit charge.

For the electric potential energy of a charge q_0 in this potential field, we can simply find:

$$U = q_0 V\left(r\right) = \frac{1}{4\pi\epsilon_0}\frac{q_0 q}{r}.$$

As such, the total potential energy of a system of point charges is:

$$U = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}}.$$

2.1 Poisson's Equation

We can take our differential form of Gauss' Law:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{\rho}{\epsilon_0}.$$

And substitute $\boldsymbol{E} = -\boldsymbol{\nabla} V$ to obtain Poisson's Equation:

$$\boldsymbol{\nabla}^2 V = -\frac{\rho}{\epsilon_0}.$$

In an area with no charge, this reduces to:

$$\boldsymbol{\nabla}^2 V = 0.$$

The $\boldsymbol{\nabla}^2$ operator, or the divergence of the gradient, is called the Laplacian. Intuitively, around a negative point charge, the potential is negative, but increases when moving away in any direction. This results in a positive divergence of the gradient (since all gradient arrows point up the potential well, away from point charge), which in Poisson's Equation corresponds with a negative charge density at that point. The gradient of the potential is only unchanging if there is no local charge density, resulting in a 0 divergence following Laplace's Equation.

2026-01-27

Lecture 7

When a sufficient potential is reached, around $30\frac{kV}{cm}$, air can be ionized and undergo dielectric breakdown. This occurs since electrons are ripped from their molecules, accelerating in the electric field and crashing into other molecules, triggering an avalanche of charge release. A high enough potential results in a self-sustaining reaction, or a corona discharge.

3 Circuits

3.1 Capacitors

Definition 6. A capacitor stores potential energy in an electric field across two separated conductors charged $\pm Q$. The capacitance $C\,(F)$ is a measure of a capacitor's ability to store energy:

$$C = \frac{Q}{V} \quad U = \frac{1}{2}CV^2 = \frac{1}{2}QV.$$

For a parallel plates capacitor in a vacuum with $l_{plate} \gg d_{sep}$:

$$E = \frac{Q}{\epsilon_0 A} = \frac{\sigma}{\epsilon_0}, \quad C = \frac{Q}{V} = \epsilon_0 \frac{A}{d}, \quad V = Ed.$$

We note that a capacitor acts like a spring in terms of charge, with analogous spring constant $\frac{1}{C}$, and exerting an 'electric force' V proportional to the stored charge.

We can compute the energy density, or energy per unit volume stored in an electric field using relations from our capacitor:

$$u = \frac{\frac{1}{2}CV^2}{Ad} = \frac{1}{2}\epsilon_0 E^2.$$

However this is valid for any electric field in a vacuum.

Lecture 8

2026-01-29

Note that capacitors combine reciprocally in series, maintaining the same stored charge across each capacitor. Capacitors combine in summation in parallel, maintaining the same voltage drop across each capacitor. Applications of capacitors includes touch screens, flash photography, or circuit decoupling. The energy density (energy per unit volume) stored in the electric field for an electric field in a vacuum is:

We can use a dielectric, or insulator in a capacitor to:

- Keep oppositely charged plates separated;
- Increase the maximum potential difference between the plates before breakdown;
- Increase the capacitance since the dielectric becomes polarized.

Definition 7. The dielectric constant $K = \frac{C}{C_0} > 1$ is defined as the ratio with which the capacitance increases when a dielectric is introduced between the plates.

When the dielectric becomes polarized, given some constant stored charge, the potential differences and original electric field decrease by the same factor $E = \frac{E_0}{K}$. The surface charge density of the capacitor decreases due to the induced separation of charge in the dielectric 'cancelling' some charge on the plates. If σ is the initial charge density on the capacitor, and σ_i is the induced charged density on the dielectric:

$$E_0 = \frac{\sigma}{\epsilon_0} \to E = \frac{\sigma - \sigma_i}{\epsilon_0} = \frac{E_0}{K}.$$

This allows us to express the induced charged density as:

$$\sigma_i = \sigma\left(1 - \frac{1}{K}\right).$$

As K becomes large, the induced charge density approaches the initial charge density, but with E and V much smaller than their values in a vacuum.

When applying Gauss' Law to dielectrics, we must account for this altered charge density when calculating the charge enclosed by the Gaussian surface. Using our relations, we can obtain:

$$\oint K\boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q_{encl,free}}{\epsilon_0}.$$

Where our inclusion of K allows us to only consider the enclosed free charges, excluding the induced charge.

We can also define a more general permittivity $\epsilon = K\epsilon_0$, and use it to redefine our formulas with a dielectric:

$$C = KC_0 = \epsilon\frac{A}{d}, \quad u = \frac{1}{2}\epsilon E^2.$$

Example. Take two concentric spherical shells with radii r_a and r_b charged with $+Q$ and $-Q$. Using Gauss' Law, we get $E = \frac{Q}{4\pi\epsilon_0 r^2}, r_a < r < b$. Finding the potential difference by integration:

$$V_{ab} = -\int_{r_a}^{r_b} E\,dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right).$$

Using $U = \frac{1}{2}QV$, we get the stored energy as:

$$U = \frac{Q^2}{8\pi\epsilon_0}|\frac{1}{r_a} - \frac{1}{r_b}|.$$

Alternatively, we integrate the energy density found using Gauss' Law between the two shells:

$$U = \int_{r_a}^{r_b} u4\pi r^2\,dr = \int_{r_a}^{r_b} \frac{1}{2}\epsilon_0\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 4\pi r^2\,dx = \frac{Q^2}{8\pi\epsilon_0}|\frac{1}{r_a} - \frac{1}{r_b}|.$$

So our two results match. ◊

2026-02-03

Lecture 9

Polarization of a dielectric in an external field occurs due to an induced separation of charge between p^+ and e^- result in the creation of a dipole aligned with the field:

$$\boldsymbol{p} = \alpha\boldsymbol{E}.$$

Where α is a proportionality constant called the atomic polarizability. Molecules with an existing charge may have their dipoles aligned by an external electric field, also resulting in polarization. Note that in more complex molecules, we must use a polarizability tensor due to the asymmetric behaviour of molecules depending on the orientation of the field.

In uncharged molecules, we also see polarization, but via free charges rather than bound charges.

3.2 Current

Electrons are always randomly moving at high speeds inside conductors ($\sim 10^6 \frac{m}{s}$ over short distances), but only undergo a net *drift* in the presence of an external electric field. Drift velocities are on the order of $\sim 10^4 \frac{m}{s}$, but occur almost instantaneously throughout the conductor since the field $\boldsymbol{E}$ propagates at nearly the speed of light.

Note. In electrical systems, we consider conventional current, or the motion of positive charges, even though this is not what occurs. △

We define the current through a conductor of cross sectional are A to be the charge flowing through that area:

$$I = \frac{dQ}{dt}(A).$$

In terms of the drift velocity v_d of n conductors of charge q per unit volume, the charge per unit time is:

$$I = qnAv_d.$$

The current density, or current per unit area is:

$$J\left(\frac{C}{m^2}\right) = \frac{I}{A}, \quad \boldsymbol{J} = nq\boldsymbol{v}_d.$$

This sums in superposition for multiple types of charge carriers in a material.

We can define the local material property of resistivity in terms of these values:

$$\rho\left(\frac{\Omega}{m}\right) = \frac{E}{J}.$$

As a function of temperature:

$$\rho(T) = \rho_0\left[1 + \alpha(T - T_0)\right].$$

Where ρ_0 is the reference temperature measured at T_0, and α is the temperature coefficient of resistivity. In some semiconductors, α is negative. Some materials have a resistivity that drops to 0 below a critical temperature T_c, called superconductivity.

The inverse of resistivity, conductivity $\sigma = \frac{1}{\rho}$ of metals is much much greater than that of insulators, so we can create circuits.

Lecture 10

2026-02-05

Inside a conductor, $E = \frac{V}{L} = \rho\frac{I}{A}$, so we can define a new quantity describing the element's *resistance*:

$$R(\Omega) = \frac{V}{I} = \frac{\rho L}{A} = \int_0^L \frac{\rho}{A(L)}\,dl.$$

We can create a temperature relation identical to our resistivity relation:

$$R(T) = R_0\left[1 + \alpha(T - T_0)\right].$$

Assuming the dimensions of the conductor do not change significantly. A resistor is a device used to dissipate energy with a controlled resistance value indicated by coloured bands, it is a linear element. Resistors can be combined as:

$$R_{eq,series} = \sum R_i, \quad \frac{1}{R_{eq,parallel}} = \sum \frac{1}{R_i}.$$

A complete circuit requires a closed loop to prevent accumulation of charge in conductors producing a cancelling internal electric field. A source of electromotive force, or potential difference, is required to drive current, doing work $W_n = qV$ on moved charges.

2026-02-10 Lecture 11

This work can be transferred into (loads) or out of (sources) an element. It is negative if the element does work, like a source.

Taking the time derivative of this change in energy, we get the power dissipated by an element:

$$P\,(W) = IV.$$

A real source of emf ϵ has a terminal voltage $V_{ab} = \epsilon - Ir$, where r is the internal resistance of the source. The output power that the source actually delivers to the load can then be found as:

$$P = I\epsilon - I^2 r.$$

Theory of Metallic Conduction

Treating electrons as classical particles with a metallic model of bonding, they are free to move around randomly and collide. Under the influence of a bias, or electric field, the motion will result in a net drift of charges. With a mean free time between collisions of τ, the net drift $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{a}\tau = \boldsymbol{a}\tau$ since on average, $\boldsymbol{v}_0 = 0$. The acceleration of the particle is $\boldsymbol{a} = \frac{q\boldsymbol{E}}{m}$. We can use this expression for the drift velocity in other equations:

$$\begin{aligned} \boldsymbol{J} &= nqv_d = \frac{nq^2\tau}{m}\boldsymbol{E} \\ \boldsymbol{J} &= \frac{1}{\rho}\boldsymbol{E} \implies \rho = \frac{m}{ne^2\tau}. \end{aligned}$$

As the temperature increases, ions vibrate and the mean free time decreases, increasing the resistivity.

Alternatively, in semiconductors, the concentration of charge carriers n increases with temperature faster than τ decreases, which results in a net increase in conductivity. In a superconductor, $\tau \to \infty$ resulting in zero resistivity.

2026-02-12 Lecture 12

The *d'Arsonval galvanometer* used a coil of wire around a permanent magnet, with a torsional spring acting as an axle for the coil. The current through the coil caused a proportional torque against the linear spring, which could be used to display a measurement on a gauge. A full scale deflection (about 90°) limited the current measurement I_{fs}, characterized by the max current

($< 10mA$) and resistance R_c of the wire. The max voltage measurement V_{fs} is related to these parameters as:

$$V_{fs} = I_{fs} R_c.$$

An ideal ammeter has 0 resistance. A galvanometer can be turned into an ammeter using a small-value shunt resistor R_{sh} in parallel with the coil. The full scale current therefore increases, as the coil only experiences a portion of the total current. The maximum voltage measurement is limited by the voltage drop across the shunt resistor when the coil is at a full-scale current measurement.

$$V_{fs} = I_{fs} R_c = \left(I_a - I_{fs}\right) R_{sh}.$$

An ideal voltmeter has infinite resistance. A galvanometer can become a voltmeter using a resistor R_s in series with the coil. A full scale reading is:

$$V_V = I_{fs} \left(R_c + R_s\right).$$

Where typically, $R_s \gg R_c$.

Lecture 13

2026-02-24

We can use the capacitor relationships, particularly $i_c = C\frac{dv_c}{dt}$, to describe time-dependent response in circuits. In this process, it is important to find initial conditions and final conditions to apply when solving the differential equation. For a series RC circuit, charging obeys:

$$i\left(t\right) = I_0 e^{-\frac{t}{RC}}, \quad \tau = RC.$$

Where $\tau = RC$ in (s) is the time constant for the circuit, corresponding with its characteristic timescale of response.

In NA, AC power comes as a 'hot' wire ($\sim 120V$) and a neutral wire. For power distribution systems, devices are connected in parallel. This allows the failure of a given device to not impact others on separate branches.

The total current drawn by a network is characterized by the power rating of devices, and the voltage at which they are driven. However, the maximum current is limited by the gauge of the wire (e.g. 12 gauge carries $\sim 20A$), and is protected by fuses or circuit breakers. After tripping, fuses must be replaced, while circuit breakers can be reset. This stops wire from overheating, as they do in a short circuit.

Note. Fuses, switches, and circuit breakers should only be placed on the 'hot' side of the line. △

Additionally, and internal grounding wire can be used to ground the frame of a device, so that if the hot wire touches it, it shorts and trips the system protection. Otherwise, the frame would become dangerous.

4 Magnetism

Magnets create a magnetic field, with field lines emanating from the North pole and continuing through the South pole. It points in the direction a test North pole would orient. Note that magnetic monopoles are not physical.

We can visualize these fields with field line diagrams, where similar to electric fields, magnetic fields never cross. However, these field lines are not line of force.

Also note that there is a unification between magnetism and moving charges. Charges create electric fields that act on other charges. Moving charges also generate a magnetic field that acts on any magnets, ferromagnetic objects, and other moving charges.

2026-03-03

Lecture 14

The magnetic force on a charged particle moving in a β field is given by:

$$\boldsymbol{F}_b = q\boldsymbol{v} \times \beta.$$

Where $\beta\,(T)$ is the vector magnetic field measured in the Tesla $1T \equiv 1\frac{N}{A\cdot m}$. Alternatively, we can use the Gauss $1G = 10^{-4}T$. Observe that this creates a force orthogonal to both the motion of the charged particle, and the direction of the field. The direction of the resulting force can be obtained via the right hand rule for the cross product.

Togther, the electric and magnetic field on a charged particle is called the Lorentz force:

$$\boldsymbol{F}_{Lorentz} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \beta\right).$$

Example. For a charged particle moving with speed v perpendicular to a magnetic field:

$$F_c = m\frac{v^2}{r} \implies q\beta = m\omega \implies \omega = \frac{q\beta}{m}.$$

Where ω is called the *cyclotron frequency.* Note that any parallel velocity to the field results in helical motion. ♢

Non-uniform magnetic fields are more complex, such as those used to confine plasmas or that of the Earth's magnetosphere (creating aurorae). Magnetic fields are used in cloud chambers, and were used in Thomson's charge-to-mass ratio of the electron experiment. They can be used in mass spectrometers, which bend particles of equal $\frac{q}{m}$ ratios (after filtering with electric and magnetic fields) at different radii depending on their mass.

We define magnetic flux to be the amount of magnetic field passing perpedicularly through a given area:

$$\Phi_B = \int_D \boldsymbol{B}\cdot d\boldsymbol{A}.$$

Where $\Phi_B\,(Wb)$ is the magnetic flux, with units webers $1Wb \equiv 1T \cdot m^2 = 1\frac{N\cdot m}{A}$. Manipulating this relationship, $\boldsymbol{B} = \frac{d\Phi_B}{dA}$ for some area element perpendicular to the flux. As such, the magnetic field is the *magnetic flux density.*

Definition 8. Gauss' law for magnetic fields states that the magnetic flux through a closed surface is always zero.

$$\oint \boldsymbol{B} \cdot d\boldsymbol{A} = 0.$$

This implies that magnetic field lines always form closed loops, and as such, magnetic monopoles cannot exist.

Lecture 15

2026-03-05

Using our Lorentz force equation for current flowing in a wire:

$$d\boldsymbol{F}_{wire} = dq\boldsymbol{v} \times \boldsymbol{B} \implies d\boldsymbol{F}_{wire} = Id\boldsymbol{l} \times \boldsymbol{B}.$$

Where $d\boldsymbol{l}$ points in the direction of the current in the wire.

Example. A straight $1m$ copper rod with $I = 50A$ in the $+x$ direction lies in a $1.2T$ magnetic field oriented 45 degrees counter-clockwise from the x-axis, find the force.

Using our differential relation, we can integrate:

$$\boldsymbol{F} = \int_0^1 I \cdot B \sin 45 \cdot \, dl \, (\hat{x} \times \hat{y}) = \frac{IB}{\sqrt{2}} (1 - 0) \, \hat{k} = \frac{60}{\sqrt{2}} \hat{k} N.$$

Where the force points along the positive x-axis. ♢

For a loop of wire in a uniform magnetic field, the net force is 0, however, the net torque may not be. In a non-uniform field, the net force will not identically be 0.

Definition 9. The magnetic moment of a closed current loop μ is defined as:

$$\boldsymbol{\mu} = I\boldsymbol{A}.$$

Where I is the current and $\boldsymbol{A}$ is the area vector, defined using the right hand rule, where fingers curl (following negative current) and the thumb points in the direction of the vector.

Note. For a solenoid, which is many loops of wire, this definition must be modified to:

$$\boldsymbol{\mu}_{\text{solenoid}} = NI\boldsymbol{A}.$$

Where N is the number of turns. △

Using the magnetic moment of the loop, the torque is given by:

$$\tau = \boldsymbol{\mu} \times \boldsymbol{\beta}.$$

Similarly, we can define a potential energy as:

$$U = -\boldsymbol{\mu} \cdot \boldsymbol{\beta}.$$

Note. We have a stable equilibrium when $\boldsymbol{\mu}$ and $\boldsymbol{\beta}$ are parallel, and an unstable when the vectors are antiparallel. △

Example. Magnets work due to many small molecular magnetic moments that may be temporarily or permanently aligned to result in a large net magnetic moment. ♢

Example. Motors work using a loop of wire in an external mangetic field. A commutator, or two brushed terminals, flips the direction of the current every half turn to fuel continuous propulsion. ♢

The magnetic force on charge carriers in a conductor travelling through an external field creates a separation of charges, and therefore an induced voltage from the top to the bottom fo the conductor. This gives:

$$n = -\frac{J_x B_y}{qE_e}.$$

Where n is the volume concentration of charge carriers, J_x is the current density in the x-direction, B_y is the magnitude of the magnetic field in the y-direction, and E_e is the electric field created across the conductor by the separation of charges in the z-direction.

2026-03-10

Lecture 16

Definition 10. The magnetic field produced by a point charge moving with constant velocity is given by:

$$\boldsymbol{\beta} = \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v} \times \hat{r}}{r^2}.$$

Where $\mu_0 = 4\pi \cdot 10^{-7} \approx 1.26 \cdot 10^{-6} T \cdot m/A$, and $\hat{r}$ point from the charge to the point at which the field is being measured.

Alternatively, we can define the field direction using the right hand rule for positive charges, and the left hand rule for negative charges, where the thumb points along the velocity vector of the charge, and fingers curl in the direction of the field.

Note. The product of the magnetic permeability and the vacuum permittivity yields:

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

△

Given a current carrying element, each of the moving charges creates a superimposing magnetic field:

$$d\boldsymbol{\beta} = \frac{\mu_0}{4\pi}\frac{dQ\boldsymbol{v}_d \times \hat{r}}{r^2}.$$

Where $dQ\boldsymbol{v}_d = Id\boldsymbol{l}$

Definition 11. The magnetic field created by a current-carrying conductor is given by the Biot-Savart law:

$$\boldsymbol{\beta} = \int \frac{\mu_0}{4\pi}\frac{Id\boldsymbol{l} \times \hat{r}}{r^2}.$$

For an infinite wire, the magnitude collapses as:

$$\beta = \frac{\mu_0 I}{2\pi r}.$$

Example. Imagine a single loop of wire with radius a on the yz plane carrying a current I counterclockwise when viewed from the positive x-axis.

The differential field element along the x-axis (since other components will cancel by symmetry) for some $d\boldsymbol{l}$ of wire is:

$$d\boldsymbol{\beta}_{1x} = \frac{\mu_0}{4\pi}\frac{Id\boldsymbol{l}}{a^2 + x^2} \cdot \frac{a}{\sqrt{a^2 + x^2}}\hat{x}.$$

Using $dl = ad\theta$ and integrating from $0 \to 2\pi$:

$$\begin{aligned}\boldsymbol{\beta}_{1x} &= \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} a\frac{d\theta}{(a^2 + x^2)^{\frac{3}{2}}} \\ &= \frac{\mu_0 I a^2}{2\left(a^2 + x^2\right)^{\frac{3}{2}}}.\end{aligned}$$

Note that the direction, for can be found using the right hand rule (conventional current), where fingers curl in the direction of current, and the thumb points in the direction of the generated magnetic field.

If the coil involved N coils of wire, we can simply multiply $N\boldsymbol{\beta}_{1x}$. At $x = 0$, i.e. in the center of the coils, the field is $\boldsymbol{\beta}_x = \frac{\mu_0 NI}{2a}$. Recall that the magnetic moment of a coil is $\mu = NIA = 2\pi NIa^2$, pointing in the direction of the field, which can be used to simplify our expression. ♢

Definition 12. Ampere's law states that the line integral of the magnetic field along an arbitrary closed path is given by:

$$\oint \boldsymbol{\beta} \cdot d\boldsymbol{l} = \pm \mu_0 I.$$

Where I is the net current enclosed by the loop, provided I is not time-variant. The result is positive if the loop direction follows the field direction.

Ampere's law obeys superposition of multiple charge carriers, similar to Gauss' law.

Lecture 17

2026-03-12

Some materials have microscopic, randomly-oriented current loops that generate magnetic fields. In the present of an external field, the material will *magnetize*, meaning the loops will align (to minimize potential energy of magnetic moment) and form a net field. A single election moving in a circle has a current $I = \frac{e}{T} = \frac{ev}{2\pi r}$. The magnetic moment is

$$\mu = IA = \frac{evr}{2} = \frac{e}{2m_e} L.$$

and angular momentum is $L = mvr = \frac{2\mu m_e}{e}$. Using angular momentum quantization:

$$2\pi r = n\lambda \implies L = rp = \frac{nh}{2\pi}.$$

Definition 13. The Bohr Magneton is given by the minimum magnetic moment due to angular momentum quantization:

$$\mu_B = \frac{eh}{4\pi m_e}.$$

The *magnetization* is the total magnetic moment per unit volume in a material:

$$\boldsymbol{M} = \frac{\boldsymbol{\mu}}{V}.$$

The total magnetic field in a material, which itself is in an 'aligning' magnetic field $\boldsymbol{\beta}_0$ is:

$$\boldsymbol{\beta} = \boldsymbol{\beta}_0 + \mu_0 \boldsymbol{M}.$$

If $\mu_0 \boldsymbol{M} > 0$, the field is enhanced, and the *relative permeability* $K_m > 1$. The enhancement fraction is given by the *magnetic susceptibility*:

$$K_m = \frac{\beta}{\beta_0}, \quad \chi_m = K_m - 1.$$

We can redefine a more general permeability using the vacuum permeability:

$$\mu = \mu_0 \cdot K_m.$$

Which we must use for calculations inside magnetized materials. Random thermal motions will oppose this alignment, following Curie's law:

$$M = C\frac{\beta}{T}.$$

However, some materials will exhibit an *opposing*, or anti-aligned magnetic field that results in:

$$K_m = \frac{\beta}{\beta_0} < 1, \quad \chi_m = K_m - 1 < 0.$$

Definition 14. Paramagnetic materials have $K_m > 1$. Diamagnetic materials have $K_m < 1$, and a negative magnetic susceptibility. Ferromagnetic materials have very large $K_m \sim 10^4$.

A third class of materials, *ferromagnetic* materials have many random *magnetic domains*, created by local alignment of atomic magnetic moments. In the presence of an external magnetic field, they will align, and even 'grow'. When all dipoles are aligned, the saturation magnetization is reached, i.e. K_m is limited by β_0.

Ferromagnetic materials may retain their magnetism until a reverse field of sufficient strength is applied: this is *hysteresis*, and forms a hysteresis loop. Dissipating this 'permanent' magnetism requires energy and dissipates heat. Permanent magnets will have a broad hysteresis curves, while motors will have narrow curves.

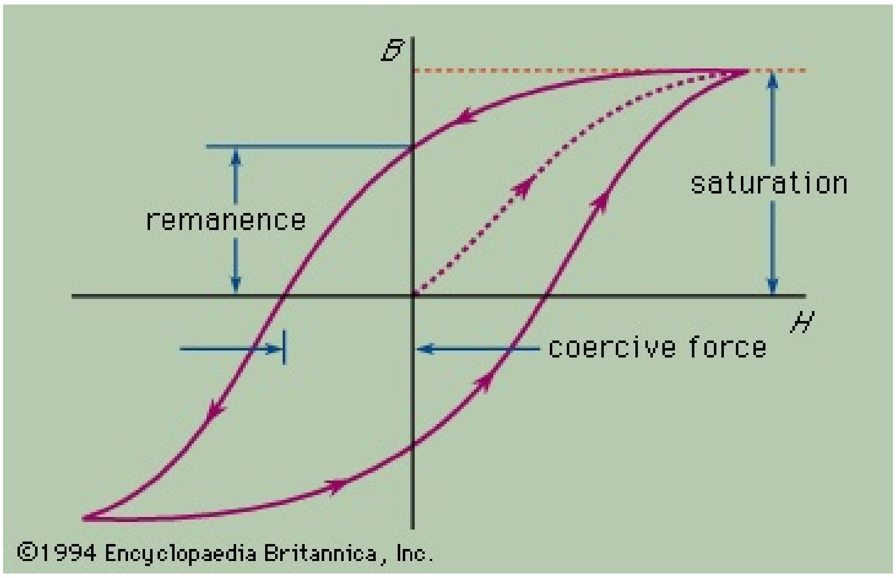

Figure 2: A hysteresis loop for a ferromagnetic material.

4.1 Induction

Similar to how we drove a motor, a change in magnetic flux through a current loop will induce an electromotive force, and therefore a current. We can induce this current by activating, deactivating, or moving the magnet, or changing the area or number of turns of the loop, etc. (change the flux Φ_B).

Definition 15. Faraday's Law states that the induced EMF is the negative time derivative of the flux:

$$\epsilon = -\frac{d\Phi_B}{dt}.$$

Note. The direction of ϵ is given by the right hand rule. It will act to create an *opposing* magnetic flux. Using the right hand rule, align the thumb opposite to the direction of increasing flux through the loop, and fingers curl in the direction of current. △

Lecture 18

2026-03-17

An alternative method to determine the direction of an induced current or voltage is via Lenz' Law.

Definition 16. Lenz' Law states that the direction of magnetic induction acts to oppose the causing change in flux. This is a consequence of the negative sign in Faraday's Law.

Example. Imagine a conducting rod moving through an external magnetic field. The magnetic force causes the charges to accumulate on one side, creating an electric field, or voltage, across the cross section of the conductor. This is the principle used in the Hall effect.

$$F_e = F_\beta \implies \epsilon = v\beta L.$$

If we were to connect the ends of this rod to a closed loop of wire, it would act as a battery of magnitude ϵ. This is a *motional EMF.* ♢

For some time-independent field β, an equivalent to Faraday's Law is:

$$d\epsilon = (\boldsymbol{v} \times \boldsymbol{\beta}) \cdot d\boldsymbol{l}.$$

Imagine a solenoid with an increasing current, causing an increasing magnetic field. A loop of wire concentric and larger than the solenoid with experience a net current travelling around the loop. This is due to an electric field throughout the loop:

$$\epsilon = \oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi_B}{dt}.$$

Where the induced electric field is non-conservative.

Electrostatic fields, generated by charged distributions, are conservative, which is *not* true for induced fields, generated by changing magnetic fields. As such, there is no associated potential function for an induced, or non-electrostatic field.

2026-03-19

Lecture 19

A time varying magnetic field acting on an extended object will induce *eddy currents.* Metal detectors take advantage of this principle by generating an oscillating magnetic field, and detecting any reciprocating magnetic fields created by the circulation of eddy currents in metal objects.

Definition 17. Recall Ampere's Law:

$$\oint \boldsymbol{\beta} \cdot d\boldsymbol{l} = \mu_0 \sum_i I_{\text{enclosed}}.$$

Where the sign of the currents is consistent with the right hand rule around the closed loop.

The symmetry between electric and magnetic fields is demonstrated by a capacitor with charging current $i_c\left(t\right)$. Taking a planar surface perpendicular to the charging wire and applying Ampere's law will allow calculation of the enclosed charging current. However, take a 'bulging surface' that shares an edge with our previous planar surface, but bulges to enclose one of the capacitor plates. There is now no longer and enclosed current, Ampere's law returns zero.

To address this, we can redefine our charging current to include the electric flux between the plates:

$$i_c\left(t\right) = \frac{dq}{dt} = \epsilon \frac{d\Phi_E}{dt}.$$

Where we define the fictitious *displacement current*:

$$i_D\left(t\right) \equiv \epsilon \frac{d\Phi_E}{dt}.$$

This allows generalization of Ampere's law as:

$$\oint \boldsymbol{\beta} \cdot d\boldsymbol{l} = \mu_0 \left(i_c + i_D\right)_{\text{enclosed}}.$$

What is most interesting about this displacement current, is that it generates a cylindrically symmetric magnetic field just like the current in the wires does! It is important to note that there is no actual current flowing between the plates, but rather a change in electric flux.

Definition 18. Maxwell's Equations are given by:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0}, \quad \text{Gauss' Law for electric fields}$$

$$\oint \boldsymbol{\beta} \cdot d\boldsymbol{A} = 0, \quad \text{Gauss' Law for magnetic fields}$$

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi_B}{dt}, \quad \text{Faraday's Law for stationary path}$$

$$\oint \boldsymbol{\beta} \cdot d\boldsymbol{l} = \mu_0 \left(i_c + i_D\right)_{\text{enclosed}}, \quad \text{Ampere's Law for stationary path.}$$

Note. We do not have to differentiate between conservative and non-conservative fields, as:

$$\oint \boldsymbol{E}_c \cdot d\boldsymbol{l} = \oint \boldsymbol{E}_n \cdot d\boldsymbol{A} = 0.$$

△

In empty space, where there are no enclosed charges or currents, we can write:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt} \oint \boldsymbol{\beta} \cdot d\boldsymbol{A}.$$

And:

$$\oint \boldsymbol{\beta} \cdot d\boldsymbol{l} = \mu_0 \epsilon_0 \frac{d}{dt} \oint \boldsymbol{E} \cdot d\boldsymbol{A}.$$

These equations show that a time varying magnetic field produces an electric field, and vice versa!

4.2 Superconductivity

For some superconductors, the critical temperature T_c decreases in the presence of an external $\boldsymbol{\beta}$, and even disappears beyond some critical field strength β_c.

Definition 19. The Meissner effect is that matter in a superconducting state ‘expels‘ all magnetic flux from it, except for possible a surface layer of ~ 100 atoms.

This effect allows for the levitation of superconductors in a strong magnetic field, where it behaves like a diamagnetic material.

4.3 Mutual Inductance

A current change in a coil of wire can induce an EMF in a neighbouring coil.

$$\epsilon_2 = N_2 \frac{d\Phi_{B_2}}{dt}.$$

Where Φ_{B_2} is the flux per coil in the second inductor. We can define the proportionality constant of the flux through the second coil induced by a current the first as the *mutual inductance*:

$$N_2 \Phi_{B_2} = M_{21} i_1 \implies \epsilon_2 = -M_{21} \frac{di_1}{dt}.$$

This is a geometric and material property. Note that mutual inductance is symmetric, so $M = M_{21} = M_{12}$, yielding:

$$M = \frac{N_2 \Phi_{B_2}}{i_1} = \frac{N_1 \Phi_{B_1}}{i_2}.$$

With $\epsilon_i = -M \frac{di_j}{dt}$.

Lecture 20

2026-03-24

Similar to mutual inductance, the self-inductance of a loop of wire describes the self-induced EMF when the current changes in the loop:

$$L = \frac{N\Phi_B}{i}.$$

Where N is the number of turns, Φ_B is the flux per turn, and i is the current. The time rate of change of the magnetic flux is then:

$$N \frac{d\Phi_B}{dt} = -\epsilon = L \frac{di}{dt}.$$

Example. Determine the self-inductance of a toroidal solenoid with a cross-sectional area A and mean radius r, closely wound with N turns of wire on a non-magnetic core. Assume $\boldsymbol{\beta}$ is uniform across a cross section.

Applying Ampere's Law for a clockwise path inside the toroid, enclosing the inner surface of current loops:

$$\oint \boldsymbol{\beta} \cdot d\boldsymbol{l} = 2\pi r\beta = \mu_0 N i \implies \beta = \frac{\mu_0 N i}{2\pi r}.$$

Finding the inductance:

$$L = \frac{N\Phi_B}{i} = N^2 \frac{\mu_0 A}{2\pi r}.$$

♢

A time-varying current through an inductor results in an induced non-conservative electric field E_n:

$$\oint \boldsymbol{E}_n \cdot d\boldsymbol{l} = \epsilon = -L\frac{di}{dt}.$$

Since the field only exists inside the inductor, and the inductor has a very small resistance, the conservative field must be roughly equal and opposite to the non-conservative field, yielding:

$$\int_a^b \boldsymbol{E}_c \cdot d\boldsymbol{l} = L\frac{di}{dt} = V_a - V_b.$$

The energy stored in an inductor is:

$$U = \frac{1}{2}LI^2.$$

The magnetic energy density stored in the inductor is:

$$u = \frac{\beta^2}{2\mu}.$$

2026-04-02

Lecture 21

Transformers

In a transformer, a primary wound around a core of very high magnetic permeability K_m transfers changes in flux to a secondary, following:

$$\frac{\epsilon_1}{\epsilon_2} = \frac{N_1}{N_2}.$$

Since the flux per turn in each coil is the same. If the windings have zero resistance, then we can relate:

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}.$$

Since power is conserved:

$$V_1 I_1 = V_2 I_2 \implies \frac{V_1}{I_1} = \frac{R}{\left(\frac{N_2}{N_1}\right)^2}.$$

Where R, the resistance in the secondary circuit is related to the resistance felt by the primary circuit, proportional to the turn ratio squared.

5 Maxwell's Equations

Recall our integral forms for Maxwell's equations presented earlier. We can convert these into differential forms.

Definition 20.

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{E} &= \frac{\rho}{\epsilon_0}, \quad \text{Gauss' Law for electric fields}\\ \boldsymbol{\nabla}\cdot\boldsymbol{\beta} &= 0, \quad \text{Gauss' Law for magnetic fields}\\ \boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{\partial\boldsymbol{\beta}}{\partial t}, \quad \text{Faraday's Law}\\ \boldsymbol{\nabla}\times\boldsymbol{\beta} &= \mu_0\left(\boldsymbol{j}_c + \epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right), \quad \text{Ampere's Law.}\end{aligned}$$

In empty space, where there are no currents or enclosed charges, we can write two simplified equations:

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{\partial\boldsymbol{\beta}}{\partial t}\\ \boldsymbol{\nabla}\times\boldsymbol{\beta} &= \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}.\end{aligned}$$

These two equations imply that a time varying field of one variety will induce a field of the other, predicting electromagnetic radiation! They yield two wave equations:

$$\begin{aligned}0 &= \frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} - \boldsymbol{\nabla}^2\boldsymbol{E}\\ 0 &= \frac{1}{c^2}\frac{\partial^2\boldsymbol{\beta}}{\partial t^2} - \boldsymbol{\nabla}^2\boldsymbol{\beta}.\end{aligned}$$

These equations show that accelerating charges create electromagnetic radiation.

A simple case is harmonic motion, which radiates power per solid angle given by:

$$\frac{dP}{d\Omega} = \frac{1}{16\pi^2\epsilon_0 c^3}q^2|\boldsymbol{a}|^2\sin^2\theta.$$

Integrating over the 2π azimuthal ϕ and the $[0,\pi]$ polar θ, where the solid angle differential element is equal to $d\Omega = \sin\theta d\theta d\phi$ yields a total power of:

$$P = \frac{q^2|\boldsymbol{a}|^2}{6\pi\epsilon_0 c^3}.$$

Note that no field perturbations are created along the axis of oscillation, and output power is maximized perpendicular to the axis of oscillation, where $\theta = 90^\circ$.

While EM waves can be produced by accelerating charges (bremsstrahlung, synchrotron radiation), they can also be produced via light-matter interaction (absorption spectra). For example, blackbody radiation is a combination of the above phenomena.

The dispersion relation for light is:

$$c = f\lambda = \frac{\omega}{k}.$$

Consider a plane wave travelling in the $+x$ direction. For $x < x_0$, there exists a uniform electric field in the $+y$ direction, and a uniform magnetic field in the $+z$ direction. For $x > x_0$ beyond the plane wave, there are no fields. This plane wave is said to be linearly polarized since the fields propagate with respect to a fixed axis. This configuration satisfies Gauss' Laws:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = 0, \quad \oint \boldsymbol{\beta} \cdot d\boldsymbol{A} = 0.$$

Provided that the wave is transverse.

Now consider Faraday's Law of induction.

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi_B}{dt}.$$

Construct a rectangular path parallel to the xy plane. The two segments parallel to the x-axis have no parallel electric field component. Set one of the vertical components of length a to be ahead of the plane wave, and the other behind. Traverse the loop counterclockwise, viewed from the $+z$ axis. We get:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -Ea.$$

Evaluating the time derivative of magnetic flux, obeying the right hand rule using the path direction we defined earlier, we get:

$$-\frac{d\Phi_B}{dt} = -\beta ac.$$

Together, these are satisfied if $E = \beta c$.

Finally, consider Ampere's Law:

$$\oint \boldsymbol{\beta} \cdot d\boldsymbol{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}.$$

Construct a rectangular path parallel to the xz plane. The two segments parallel to the x-axis have no parallel magnetic field component. Set one of the components parallel to the z-axis of length a to be ahead of the plane wave, and the other behind. Traverse the loop counterclockwise, viewed from the $+y$ axis. We get:

$$\oint \boldsymbol{\beta} \cdot d\boldsymbol{l} = \beta a.$$

Evaluating the flux time derivative (following the right hand rule using our defined path direction):

$$\mu_0 \epsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \epsilon_0 Eac.$$

Setting these equal and using our Faraday's Law result, we get:

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

2026-04-07

Lecture 22

Imagine the wave was no longer propagating in vacuum. We would obtain the modified relations:

$$E = v\beta$$
$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{\sqrt{KK_m}}.$$

Where $\epsilon = K\epsilon_0$ and $\mu = K_m\mu_0$. Since $K_m \sim 1$ and $K > 1$, the speed of light always decreases by some ratio defined as the index of refraction:

$$n = \sqrt{KK_m} \approx \sqrt{K}.$$

Also note that K is as function of frequency, meaning that so is the index of refraction! This is because at higher frequencies, the electric dipoles in a material cannot flip quickly enough, changing the K experienced.

We discovered 4 critical points:

1. The wave is bi-transverse, meaning the direction of propagation $\hat{k}$ point in the direction of the cross product:
$$\hat{k} \propto \boldsymbol{E} \times \boldsymbol{\beta}.$$
2. The defined ratio of the electric and magnetic field magnitudes is:
$$E = c\beta.$$
3. The waves travel with a defined speed:
$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}}.$$
4. The waves can travel in vacuum

For a sinusoidal EM wave travelling in space in the positive x direction:

$$E_y(x,t) = E_{max}\cos(kx - \omega t)$$
$$\beta_z(x,t) = \beta_{max}\cos(kx - \omega t).$$

Where $\omega = 2\pi f$ is the angular frequency and $k = \frac{2\pi}{\lambda}$ is the wave vector. Note that the electric and magnetic fields oscillate in phase!

5.1 Energy of Waves

Since EM waves carry energy, and are composed of electric and magnetic fields, we can work with the superimposed energy densities:

$$u = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}\beta^2 = \epsilon_0 E^2.$$

Where our speed of light relation shows that the two fields have equivalent energy densities. From this, the differential energy element travelling through some area is:

$$dU = \epsilon_0 E^2 Acdt.$$

So the energy per unit time per unit area is:

$$S = \epsilon_0 c E^2 = \frac{E\beta}{\mu_0}.$$

Defining this as a vector pointing in the direction of energy transfer, or wave propagation (the Poynting Vector):

$$\boldsymbol{S} = \frac{1}{\mu}\boldsymbol{E} \times \boldsymbol{\beta}.$$

The instantaneous power through a given surface is then:

$$P = \int \boldsymbol{S} \cdot d\boldsymbol{A}.$$

For a sinusoidal wave, we have:

$$S\left(x,t\right)\hat{x} = \frac{E_{max}\beta_{max}}{\mu_0}\cos^2\left(kx - \omega t\right).$$

Averaging over one cycle, we get the intensity of the EM wave in vacuum:

$$\left\langle S\left(x,t\right)\right\rangle = \frac{E_{max}\beta_{max}}{2\mu_0} = \frac{E_{max}^2}{2\mu_0 c} = \frac{1}{2}\epsilon_0 c E_{max}^2.$$

Let's define the momentum per unit volume of EM waves:

$$\frac{dp}{dV} = \frac{E\beta}{\mu_0 c^2} = \frac{S}{c^2}.$$

The differential element of momentum for some area is then:

$$dp = \frac{S}{c^2} \cdot Acdt \implies p = \frac{A}{c}\int S\left(x,t\right) \cdot dt = \frac{A}{c}\frac{E_{max}\beta_{max}}{\mu_0}\int \cos^2\left(kx - \omega t\right)dt.$$

We can evaluate this as:

$$p = A\frac{E_{max}^2}{\mu_0 c^2}\left[\frac{t}{2} - \frac{\sin\left(4kx - 4\omega t\right)}{4\omega}\right].$$

Averaging the momentum for a sinusoidal wave:

$$\langle p \rangle = At\frac{E_{max}^2}{2\mu_0 c^2} = At\frac{S_{av}}{c}.$$

Finding the time-averaged radiation pressure of our sinusoid as $P = \frac{F}{A} = \frac{dp}{dt} \cdot \frac{1}{A}$:

$$\langle p_{rad} \rangle = \frac{S_{av}}{c} = \frac{I}{c}.$$

Where $I \equiv$ intensity. This assumes that the light is absorbed. If the light was reflected, then the impulse is doubled:

$$\langle p_{rad} \rangle = 2\frac{I}{c}.$$

Lecture 23

2026-04-09

EM waves can be reflected by conductors or dielectrics.

All incident perpendicular $\boldsymbol{E}$ fields upon an ideal conductor are instantly nullified due to the movement of charges. This creates an $|\boldsymbol{E}| = 0$ boundary condition. This movement of charges produces an equal and opposite electric field, which cancels the field upon the conductor surface, but also radiates a reflection of the incident wave.

We can model this with:

$$\begin{aligned} E_y\left(x,t\right) &= E_{\max}\left[\cos\left(kx+\omega t\right) - \cos\left(kx-\omega t\right)\right] \\ \beta_z\left(x,t\right) &= \beta_{\max}\left[-\cos\left(kx+\omega t\right) - \cos\left(kx-\omega t\right)\right]. \end{aligned}$$

Using trig identities, we get:

$$\begin{aligned} E_y\left(x,t\right) &= -2E_{\max}\sin\left(kx\right)\sin\left(\omega t\right) \\ \beta_z\left(x,t\right) &= -2\beta_{\max}\cos\left(kx\right)\cos\left(\omega t\right). \end{aligned}$$

For E, we have nodal planes at $x = 0, \frac{\lambda}{2}, \lambda, \ldots$, and for β, we have nodal planes at $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \ldots$. Note that the nodal planes are not in phase, but are both separated by $\frac{\lambda}{2}$. If we insert a second conducting plane at one of the electric field nodal planes, we will create a standing wave!

Note. A second conducting plane at a non-nodal location results in no allowed standing waves for a given frequency. △

Both conducting plates must align with nodal planes:

$$L = n\frac{\lambda}{2}, n = 1, 2, 3, \ldots \in \mathbb{Z}^+.$$

This results in allowed frequencies:

$$f = \frac{cn}{2L}.$$