Introduction to Modern Physics

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Contents

1 The Experimental Basis of Quantum Mechanics

2026-01-05

Lecture 1: Historical Introduction

Before 1895, Newtonian mechanics seemed to reign supreme over everything (well, mechanics, electromagnetism, and thermodynamics).

- Understand atomic theory, why atoms have distinct characteristics
- Show luminiferous ether as the medium of EMR in an experiment
- Show that energy was a continuously variable quantity (no quantized energy)

In the following years, the 1895 discovery of the x-ray, the 1897 discovery of the electron (CRT, e/m ratio, quantized charge), and Planck's 1900 blackbody radiation model $E = hf$ led to the birth of quantum physics. 1905: Einstein publishes 3 papers on the photoelectric effect (idea of a photon), Brownian motion (existence of atoms), and special relativity. From 1895 onwards, we delve into the world of modern physics.

2026-01-07

Lecture 2: The Experimental Basis of Quantum Mechanics

Discovery of the X-ray and the electron

Crooke's tubes were used to generate cathode rays. In 1895, Wilhelm Rontgen produced a new 'X-ray' from the collision of the cathode rays with the anode, discovering that it could image bones through his hand. This was used for medical applications immediately!

In 1897, J.J. Thomson discovered that cathode rays were constituted by negative particles of charge $-e$ with a given $\frac{q}{m}$.

Definition 1. The Lorentz force is the combination of the electric force and magnetic force acting on a *signed* charged particle:

$$\boldsymbol{F} = \boldsymbol{F}_E + \boldsymbol{F}_B = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}.$$

We can use the RHR (for + charges) or the LHR to determine the direction of the force. Align the thumb with the direction of motion, and the fingers in the direction of the field, and the palm will 'push' in the direction of the force.

If the forces are cancelled, we obtain:

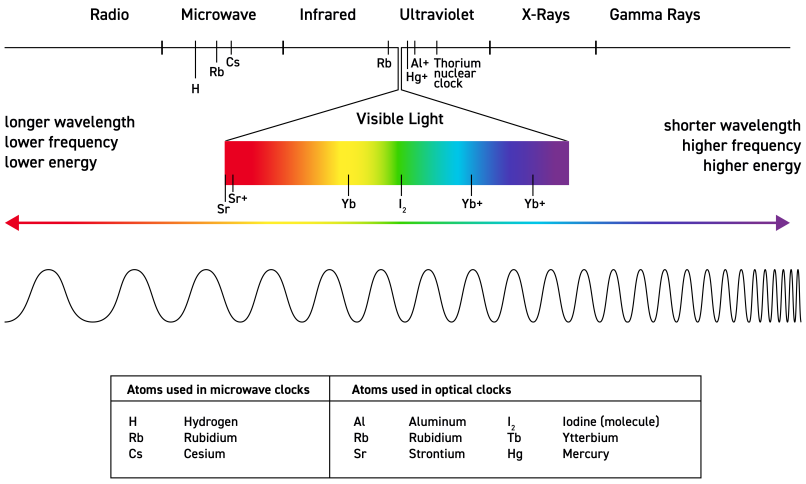
$$F_E = F_B \implies -eE = -evB \implies v = \frac{E}{B}.$$

If the Lorentz force is strictly magnetic, a moving charged particle will move in uniform circular motion:

$$F_{centripetal} = evB = \frac{mv^2}{R} \implies \frac{e}{m} = \frac{v}{BR}.$$

Determination of the electron charge

In 1909, Millikan completed his oil drop experiment, involving many trials, to obtain an experimental value for the elementary charge $e = -1.6022 \cdot 10^{-19} C$. Using Thomson's $\frac{e}{m}$ ratio, he also determined a value for the mass of the electron, and found the ratio of the H-atom to electron mass of $\frac{m_p}{m_e} \approx 1837$.



Atoms used in microwave clocks		Atoms used in optical clocks			
H	Hydrogen	Al	Aluminum	I_2	Iodine (molecule)
Rb	Rubidium	Rb	Rubidium	Tb	Ytterbium
Cs	Cesium	Sr	Strontium	Hg	Mercury

Figure 1: The electromagnetic spectrum. Recall that a green laser has a wavelength of $\lambda = 532nm$.

Lecture 3

2026-01-09

1.1 Quantization

In 1704, Newton examined *Opticks* through the diffraction of light through a prism, and adopted a particular stance. However, when light traverses an aperture, it spreads out. A smaller aperture leads to a wider spread. In 1678, Huygens modelled these single-slit diffraction pattern with a wavelet model of light, in which the wave front is made up of sources. In 1801, Thomas Young examined double-slit interference, which was the first clear demonstration of the wave nature of light. Hans Christian Oersted discovered a connection between electricity and magnetism in 1820, improved by Faraday in 1842.

Maxwell unified these advancements into his 4 equations, that 'confirmed' the wave nature of light. His equations also predicted the speed of light in a vacuum as $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \cdot 10^8 \frac{m}{s}$. However, what do light waves travel in? Answer: the ether (wrong!).

1.2 Line Spectra

From the mid-1800s, it was known that chemical elements emit unique wavelengths of light when excited (burned or electrically activated). These spectral profiles were viewed using spectrometers made of prisms or diffraction gratings.

2026-01-12 **Lecture 4**

Light is an electromagnetic wave with transverse electric and magnetic fields. The wavelength

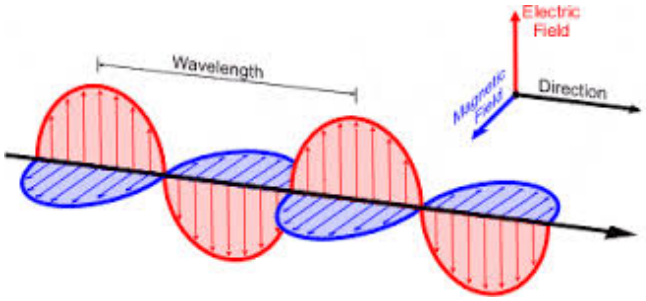


Figure 2: Light is a bi transverse wave, with oscillation perpendicular to direction of travel. Note that $\boldsymbol{E} \times \boldsymbol{B}$ points in the direction of oscillation, following RHR.

λ is the distance between two points of equal phase on either the $\boldsymbol{E}$ or $\boldsymbol{B}$ wave. In a vacuum, $c = \lambda f \approx 3 \cdot 10^8 \frac{m}{s}$.

Example. Frequency of green laser pointer

$$f = \frac{c}{\lambda} = \frac{3 \cdot 10^8 \frac{m}{s}}{532 \cdot 10^{-9} m} = 564 \cdot 10^{12} s^{-1} = 564 THz.$$

$\Diamond$

Let's examine the diffraction of light from a single slit. Any wave passing through an aperture will diffract to an extent determined by the ratio of the slit width to the wavelength. A double slit experiment uses the diffraction of light through an aperture, along with interference from the two slits to result in an interference pattern. In analysing Young's double slit, we assume the distance to the screen $L \gg d$, where d is the inter slit distance. This allows us to find the path length difference between two rays travelling from either slit to the same spot on the screen as

$$\Delta s = s_2 - s_1 = d \sin \theta.$$

Where θ is the angle of the ray from the horizontal, and is used to create a congruent right angle triangle between the two rays. If this path length is an integer multiple of the wavelength, then we will have constructive interference at that θ.

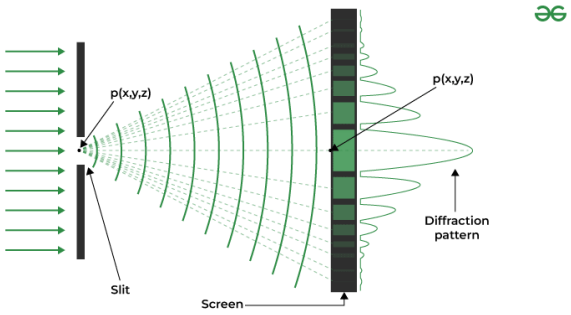


Figure 3: Single slit diffraction of planar wave. A circular aperture would create an airy disk pattern.

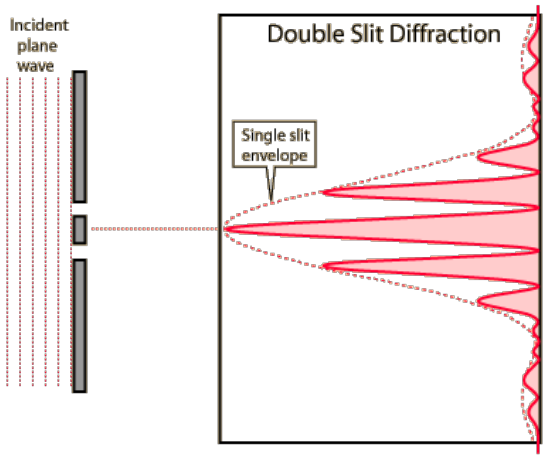


Figure 4: A double slit experiment, showing the combine effects of single slit diffraction and interference.

Definition 2. In a double slit experiment, constructive interference occurs for θ satisfying:

$$n\lambda = d\sin\theta, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots \in \mathbb{Z}.$$

And destructive interference for:

$$(n - \frac{1}{2})\lambda = d\sin\theta, \quad n \in \mathbb{Z}.$$

In a diffraction grating, provided uniform slit spacing d, we will be able to use the same formula. The orientation of the diffraction grating will diffract light perpendicular to the direction of the slits – vertical slits = horizontal diffraction.

Note. To see a diffraction pattern, $d > \lambda$ is a required condition. Comparing with our formula,

Figure 5: A diffraction spectrometer diffracts a beam of collimated light.

since sin is bounded, observe that $\lambda = d$ results in the first maxima disappearing:

$$\sin\theta = \frac{n\lambda}{d} = 1 \implies \theta = 90.$$

This means that to use things like x-ray diffraction, the resolution is limited by needing a grating with spaces larger than the wavelength. This is called an effective medium. △

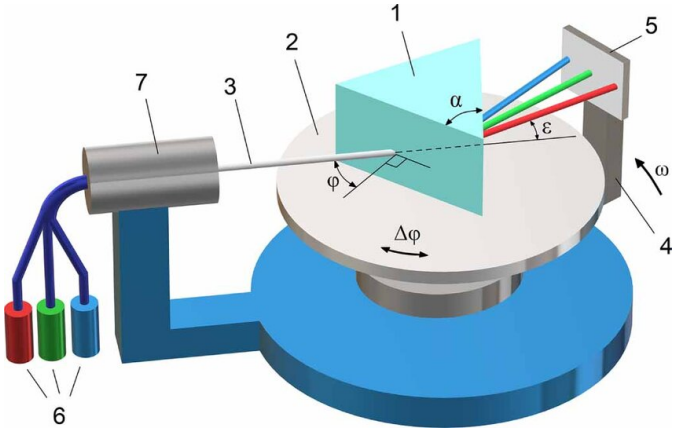


Figure 6: A prism spectrometer bends shorter wavelengths more.

2026-01-14

Lecture 5

Hydrogen Line Spectra

Balmer figured out this simplified formula for the wavelengths in the H spectrum a:

$$\lambda = 364.56\frac{k^2}{k^2 - 4}.$$

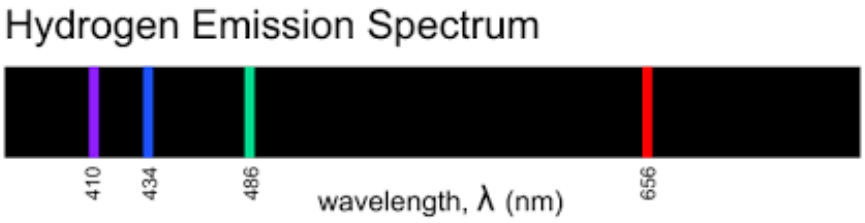


Figure 7: The Balmer series, showing hydrogen emission. There is limit for this series at $365nm$ that is too blue to see.

Where $k \in \mathbb{Z}^+, k > 2$ and $k = 3$ corresponds with the lowest energy $656nm$ emission, then $k = 4$ with $486nm$. This limit of this series is $k = \infty \to \lambda = 365nm$. We can rewrite this as:

$$\frac{1}{\lambda} = R_H \left(\frac{1}{2^2} - \frac{1}{k^2} \right).$$

Where $R_H = 1.096776 \cdot 10^7 m^{-1}$ for the Hydrogen atom characterizes the Rydberg formulation. The more general form is the Rydberg equation for the spectral line wavelengths of H.

$$\frac{1}{\lambda} = R_H \left(\frac{1}{n^2} - \frac{1}{k^2} \right).$$

Where n is the energy level that an excited electron is falling to, and characterizes the series:

- $n = 1, k > 1$ = Lyman series, UV;
- $n = 2, k > 2$ = Balmer series, UV/visible;
- $n = 3, k > 3$ = Paschen series, IR.

1.3 Blackbody Radiation

Definition 3. An ideal blackbody is able to absorb all the radiation incident on it. By Kirchoff's Law of Thermal Radiation, a body at thermal equilibrium has a rate of absorption equal to a rate of emission of radiation. Therefore, a good absorber must be a good emitter to stay in thermal equilibrium or follow the laws of thermodynamics.

Emissivity $\epsilon = 0 \to 1$, where $\epsilon = 0$ is a perfect reflector, and $\epsilon = 1$ is an ideal blackbody.

Take a surface with black and white paint. If exposed to radiation, the black surface will heat up due to a higher emissivity $\epsilon \to 1$. Note that heating up due to radiation depends on the incoming radiation, while emission depends on the body's temperature, so until equilibrium, $P_{abs} \neq P_{emit}$. However, if the two surfaces are held at the same temperature, the black surface will *still emit more radiation that the white surface*, due to its higher emissivity. It is passively absorbing more radiation, and therefore must emit more to stay in thermal equilibrium.

A box with a hole is a good blackbody, $\epsilon \to 1$. The hole will even appear darker than a black-painted surrounding surface when looking at it due to many reflections causing absorption of light in the blackbody cavity. The hole therefore reflects less, or absorbs more, light than the paint. Since both the hole and the surface are in thermal equilibrium, the hole radiates more energy per unit area to maintain net 0 energy flow. If heated, following the Stefan-Boltzmann

law $R = \epsilon\sigma T^4$, the hole's emissive power per unit area increases faster than the paint's due to the higher emissivity $\epsilon \to 1$ of the hole. It glows disproportionately brighter than the paint. Therefore, it will appear brighter on thermal imaging in contrast to the background absorption from the surroundings, which are at a lower temperature.

We can use a diffraction grating to view the spectral output of a blackbody. The wider the slit, the more energy we read, but the less wavelength resolution we get.

As we heat up the black body, the peak of the spectral energy curve shifts to shorter and shorter wavelengths, and the whole curve gets taller. No one could explain this behaviour, until in 1900, Max Planck discovered that this could be modelled if the energy of the EM waves was quantized.

2026-01-16

Lecture 6

- Spectral energy $U\,(J)$ is the total energy contained in radiation over a given wavelength range in a given volume.
- Spectral energy density $u\left(Jm^{-1}\right)$ is the spectral energy per unit wavelength range in a volume:

  $$dU = u\,(\lambda)\,d\lambda.$$

  We typically plot $u\,(\lambda)$ against λ, sampled using a diffraction grating on blackbody emissions. This can be integrated for U in a certain (λ_1, λ_2) range.
- Spectral intensity, or really total emissive power, $I\left(\frac{W}{m^2}\right)$ or R can be obtained by integrating I_λ over the wavelength range
- Spectral intensity density, or really spectral emissive power $I_\lambda\left(\frac{W}{m^3}\right)$ or d is the spectral intensity per unit wavelength range:

  $$dR = I_\lambda d\lambda.$$

- Power $P = IA = RA$, where A is area of emitting surface and R is the total emissive power.

Definition 4. Wein's displacement law describes the wavelength at the peak of the blackbody emission spectrum:

$$\lambda_{max} T = 2.898 \cdot 10^{-3} m \cdot K.$$

Where $T\,(K)$ is the blackbody's temperature.

Definition 5. The Stefan-Boltzmann law is another phenomenological law that describes the emissive power (power per unit area) of the blackbody.

$$R = \epsilon\sigma T^4.$$

Where R is the spectral intensity, ϵ is the emissivity, σ is the Stefan-Boltzmann constant, and $T\,(K)$ is the temperature. Intensity $\propto T^4$.

Example. What is the peak emission wavelength of a human body, $T = 37C = 310.15K$.

Using Wein displacement law:

$$\lambda_{max} = 2.898 \cdot 10^{-3} m \cdot K \cdot \frac{1}{310K} = 9.35 \cdot 10^{-6} m = 9.35 \mu m.$$

This is nid-infrared, and is why thermal imagers lie in the $9\mu m$ sensitivity range. ◊

Example. Peak emission wavelength of the sun is about $500nm$. What is the temperature of the surface of the sun?

Using the Wein displacement law:

$$T = 2.898 \cdot 10^{-3} \cdot \frac{1}{500 \cdot 10^{-9}} \approx 5800K.$$

Using this, what is the power per unit area, or the spectral intensity, radiated by the sun?

Using the Stefan-Boltzmann law:

$$R = \epsilon \sigma T^4 = 1 \cdot \left(5.6704 \cdot 10^{-8} \frac{W}{m^2 K^4}\right) (5800K)^4 = 6.4 \cdot 10^7 \frac{W}{m^2}.$$

Using the radius of the sun, we get $P = 3.9 \cdot 10^{26} W$ distributed across the shell surface area.

What I do we see on Earth?

$$I = \frac{P}{A} = \frac{3.9 \cdot 10^{26} W}{4\pi \left(R_e = 1.49 \cdot 10^{11} m\right)^2} = 1398 \frac{W}{m^2} \approx 1400 \frac{W}{m^2}.$$

◊

Spectral Energy Density of a Blackbody

The blackbody is like a conducting box, which acts as a resonator for EMR. The walls enforce the boundary condition $E = 0$ at the walls. This results in allowed wavelengths (i.e. modes or standing waves). Counting modes per unit wavelength range:

$$N(\lambda)\, d\lambda = \frac{8\pi V}{\lambda^4} d\lambda.$$

At lower wavelengths, there are more modes, as we would expect for standing waves on a string. Therefore the spectral energy density is the number of modes per unit wavelength times the average energy expected associated with each mode, at a given wavelength λ:

$$u(\lambda) = \frac{dU}{d\lambda} = N(\lambda)\, \overline{E}(\lambda) = \frac{8\pi V}{\lambda^4} \overline{E}(\lambda).$$

Where V is the volume and $\overline{E}$ is the energy function.

Classical Failure

Equipartition theorem for ideal gases says each degree of freedom in the system has energy $\frac{1}{2} k_B T$. Polarization of light has two basis vectors, so we take the average energy to be:

$$\overline{E} = k_B T.$$

Therefore, our spectral energy per wavelength in our volume is:

$$\frac{8\pi V}{\lambda^4} k_B T.$$

Which implies infinite energy from a blackbody source integrated over all λ. WRONG, this is the ultraviolet catastrophe!! Our classical prediction works for longer wavelengths, but at short λ, the classical prediction diverges!

Quantum Revelation

Max Planck found that he could solve this if we restrict the energy of an EM wave to be $E = nhf, \forall n \in \mathbb{Z}^+$. This restricts the energy of a given wavelength, which also restricts the allowed amplitudes we can use for a standing wave, which limits the energy! It makes it harder to excite these high energy photons. This gives the average energy for an allowed mode as:

$$\overline{E} = \frac{hf}{\exp\left(\frac{hf}{k_B T}\right) - 1}.$$

At low frequencies, the classical model is 'recovered,' however at high frequencies or short wavelengths, the exponential term suppresses the energy of energy of the modes. Which gives a spectral energy density of:

$$u\left(\lambda\right) = \frac{8\pi V}{\lambda^5} \frac{hc}{\exp\left(\frac{hc}{k_B T \lambda}\right) - 1}.$$

More commonly, we use the spectral emissive power (power per unit area per unit wavelength) and express the Planck law in the form:

$$I_\lambda = \frac{2\pi c^2 h}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{k_B T \lambda}\right) - 1}.$$

2026-01-19 **Lecture 7**

1.4 Guest Lecture: Introduction to Cosmology

Here are some observations:

- Light emitted by elements from galaxy clusters are red shifted.

  $$z = \frac{\lambda_{obs} - \lambda_{lab}}{\lambda_{lab}}.$$

  If $z \ll 1$, we can use non-relativistic Doppler effect to fond relative speed.

- Hubble's Law: Farther galaxies have larger redshifts.

  Using IR telescopes, we can 'see' ultra-redshifted galaxies. This law can be explained by an expanding universe of galaxy clusters. Redshift occurs because of expansion of space, not because of relative speeds. An observed wavelength of $\lambda_{obs} = (1+z)\lambda_{lab} \implies$ space has expanded by $(1+z)\,L$ from time of emission. Mathematically, since its uniform and angle preserving, it is a conformal expansion.

- Hubble's Law on a cosmological scale is isotropic: it is direction-independent.

  Since we assume that we are not at the center of the universe, our viewpoint of isotropy implies that the universe is likely homogeneous (isotropic to all observers). If we run our conformal expansion backwards in time, the density of the universe increases at every possible location simultaneously.

- Olber's Paradox: the sky is dark at night despite an infinite number of galaxies.

  If the universe is not infinitely old, then the speed of light could limit our observations. NOTE: the sky is not completely dark! In the radio wave spectrum, we see *cosmic microwave background radiation* emanating from every point in the sky, at nearly isotropic levels.

- The universe is chemically simple, made primarily of H and He.

Note. A gas of photons is described by the temperature:

$$\frac{Number}{Volume} = n = C \times T^3.$$

Therefore is the universe is expanding, the photon density deceases $\implies$ the universe cools. $\triangle$

In the earlier, hotter, days of the universe, it was opaque due to the scattering of photons through plasma. An opaque hot body emits a blackbody spectrum. With the expansion of the universe, the wavelengths in the blackbody spectrum are stretched by $1+z$, and the temperature is shifted down by $\frac{T}{1+z}$. As the plasma cools below $3000K$, protons and electrons recombine into H, and the universe becomes transparent. Since this 'time of last scattering', about 300000 years after the big bang, photons now have been shifted into the microwave region.

Imaging space yields a distribution of microwave background radiation, anisotropies, from the average temperature.

Lecture 8

2026-01-21

1.5 Photoelectric effect

Definition 6. The photoelectric effect is when EMR incident on a metal liberates electrons, dubbed photoelectrons.

In 1905, Einstein found that energy quantization $E_n = nhf$ used by Planck to explain blackbody radiation is a fundamental property of light. Photon energy is given by:

$$E = hf = \frac{hc}{\lambda}.$$

Note that the product $hc = 1240nm \cdot eV$, where $1eV$ is the energy that one electron acquires across a potential difference of $1V$.

The photoelectric effect is characterized by a minimum amount of photon energy to liberate electrons, called the work function $E > \phi$. The Fermi energy is the highest occupied energy level in the metal, which at the edge of a conductor, is met with a potential wall of height $\phi \sim 4eV$. Note that increasing the photon energy (frequency) results in faster photoelectrons, while increasing the intensity (no. photons) increases the current of photoelectrons. This only occurs for $E = hf > \phi$, and no electrons are liberated otherwise. These results contradict the classical wave theory.

We can calculate the max kinetic energy of a liberated electron as $K_{max} = hf - \phi$. We can measure this by liberating electrons from an anode inside a capacitor, and measure what applied voltage stops the current. Note that two-photon photoelectric effect can sometimes be used.

Example. What is the maximum wavelength of incident light that can produce photoelectrons from silver with $\phi = 4.64eV$? What will be the kinetic energy and speed of these photoelectrons if the wavelength is halved?

$$E = \frac{hc}{\lambda} = \phi \implies \lambda = \frac{1240nm \cdot eV}{4.64eV} = 267.2nm.$$

For halved wavelength:

$$K = \frac{hc}{\lambda} - \phi = \frac{1240nm \cdot eV \cdot 2}{267.2nm} - 4.64eV = 4.64eV \implies v \approx 1 \cdot 10^6 \frac{m}{s}.$$

◊

Example. A green laser pointer with $10mW$ average power hits a screen. How many photons per second hit the screen?

The energy of the photons is $E = hf = 2.33eV$.

$$P = \frac{dE}{dt} = \frac{Nhf}{dt} \implies N = \frac{0.01\frac{J}{s}}{2.33eV \cdot 1.6 \cdot 10^{-19}\frac{C}{eV}} = 2.7 \cdot 10^{16} \frac{photons}{second}.$$

◊

2026-01-23

Lecture 9

X-rays are high energy $(1 - 100keV)$ penetrating radiation that can be generated by the rapid deceleration of an electron beam. In synchrotron radiation, this is accomplished by bending fast-moving electrons, but it can also be created with electron beams hitting an anode metal (bremsstrahlung, or braking radiation).

The maximum energy we can give to an x-ray photon in this process, $\lambda_{min} = \lambda_c$ is limited by the accelerating voltage V_0:

$$eV_0 = hf_{max} = \frac{hc}{\lambda_{min}}.$$

The wavelength of x-rays is comparable to atomic distances, meaning crystal structures can act as a diffraction grating (wider gaps than x-ray wavelengths).

Relativity Review

A particle with *rest* mass m is moving with velocity $\boldsymbol{u}$ relative to the K frame. The momentum of the particle $\boldsymbol{p} = \gamma m \boldsymbol{u}$ for γ given by:

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}.$$

The relativistic kinetic energy of the particle is $T = mc^2(\gamma - 1) = \gamma mc^2 - mc^2$. For $u \ll c$, use the Taylor expansion on γ with $(1-x)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}x$, which gives $T = \frac{1}{2}mu^2$. The total energy is:

$$E = \gamma mc^2 = T + mc^2 = T + E_0.$$

Where $E_0 = mc^2$ is the rest energy associated with a particle of rest mass m.

The rest mass of a particle can be expressed as $m = \frac{E_0}{c^2}\left(\frac{eV}{c^2}\right)$, which for an electron is $0.511\frac{MeV}{c^2}$. This gives a rest energy of $E_0 = 5.11 \cdot 10^5 eV$. For a proton, the rest mass is $m = 938.27\frac{MeV}{c^2} \implies E_0 = 938.27MeV$.

Example. A transmission electron microscope accelerates electrons through $\Delta V = 300kV$.

The kinetic energy of the electrons is $T = 300keV = (\gamma - 1)\, mc^2$.

$$\gamma = \frac{K}{E_0} + 1 = \frac{K}{mc^2} + 1 = 1.586.$$

Corresponding with a velocity of $u = 0.78c$. ◊

Note that if $K \ll E_0$, then classical formulation applies to a good approximation. This corresponds with $\gamma \approx 1$.

We can take the conservation of mass-energy:

$$E_{total} = K_i + E_{0i} = K_f + E_{0f}.$$

A classic example is two particles of equal rest mass m travelling towards each other at equal speeds in some inertial frame and combining to mass M. Observe that applying conservation of mass-energy:

$$2K + 2mc^2 = Mc^2 \implies M = 2m + \frac{2K}{c^2} = 2m\gamma > 2m.$$

Let's rearrange momentum a bit:

$$\begin{aligned} p &= \gamma m u \\ p^2c^2 &= \gamma^2 m^2 u^2 c^2 \\ (pc)^2 &= \gamma m^2 c^4 - m^2 c^4 \\ \frac{u^2}{c^2} &= 1 - \frac{1}{\gamma^2}. \end{aligned}$$

Note that in this form: $(pc)^2 = \left(\gamma mc^2\right)^2 - \left(mc^2\right)^2$, where the first term is E and the second term is the rest mass E_0. Therefore:

$$E^2 = (pc)^2 + E_0^2.$$

For a photon, $m = 0$ so the rest energy is $E_0 = 0$, so the total energy being non-zero implies that photons have momentum:

$$p = \frac{h}{\lambda}.$$

1.6 The Compton Effect

Lecture 10

2026-01-26

Definition 7. Compton scattering is the scattering of x-rays off free electrons. This obeys conservation of energy of momentum. For Compton scattering off of the electron:

$$\Delta\lambda = \frac{h}{mc}\left(1 - \cos\theta\right).$$

Where θ is the scattering angle of the electron, and ϕ is the recoil angle of the photon.

In classical EM, the radiation pressure $P = \frac{I}{c}$ relates the intensity and the speed of light. This is consistent with the novel photon momentum concept, which quantizes the radiation pressure.

Let's derive it. By conservation of momentum:

$$\boldsymbol{p}_1 = \boldsymbol{p}_2 + \boldsymbol{p}_e.$$

This gives:

$$p_e^2 = p_1^2 + p_2^2 - 2p_1p_2\cos\theta.$$

Applying conservation of energy:

$$E_1 + mc^2 = E_2 + E_e.$$

The maximum possible $\Delta\lambda = \frac{2h}{mc} \approx 0.00486nm$ for a back-scatter collision means we use x-rays to see a larger relative difference.

Example. What incident wavelength results in a 10% change in wavelength to be observed?

Given $\Delta\lambda_{max} = 0.00486nm$, and $\frac{\Delta\lambda}{\lambda_1} = 0.1$, we get:

$$E_1 = \frac{hc}{0.0486nm} = 25.5keV.$$

$\Diamond$

See notes for diagram regarding elastic (no wavelength change) and inelastic scattering (change energy of photon).

1.7 Pair Production

Definition 8. Pair production is when a gamma ray turns into an electron and a positron.

The electron and positron have equal masses, and therefore rest energies of about $511keV$, but opposite polarized charge. Energy must be conserved:

$$hf = E_+ + E_-.$$

And conservation of momentum:

$$\begin{aligned}\frac{hf}{c} &= p_-\cos\theta_- + p_+\cos\theta_+ \\ 0 &= p_-\sin\theta_- + p_+\sin\theta_+.\end{aligned}$$

This must occur near a nucleus for these equations to be possible. The nucleus momentum gets a small 'kick' that helps satisfy the equations, but it implies that:

$$hf = E_+ + E_- + K_{nucleus} \implies hf > 2m_ec^2 = 2E_0.$$

This means the photon energy must be *greater* than the rest energies of the positron and electron for pair production.

Example. Consider an electron e^- travelling right with $u = 0.9c$ and a positron travelling left at $u = -0.9c$. These particles annihilate, producing two photons of equal energy and opposite velocities to satisfy conservation of energy and momentum.

These particles have $\gamma = \frac{1}{\sqrt{1-0.9^2}} = 2.294$. Using conservation of energy:

$$E_{e-} + E_{e+} = 2E_\gamma \implies 2\gamma mc^2 = 2E_\gamma \implies E_\gamma = \gamma mc^2 = 1.49MeV.$$

$\Diamond$

Example. Antimatter meteorite question in the notes: answer is $R \approx 79km$ $\Diamond$

Lecture 11 2026-01-28

2 Structure of the atom

Questions about the atom in 1900:

- What gives atoms distinct chemical and spectral characteristics?
- What is the structure of the atom (internal, electrons, etc.)?
- What causes radioactivity?

J.J. Thomson's 'plum pudding' model proposed a homogenous, positively charged mass with electrons embedded within. His model proposed these solutions:

- Blackbody behaviour: Heating atoms results in electrons vibrating more, producing more EM radiation

Problems:

- No explanation of spectral lines, radioactivity, ...

2.1 Scattering Experiment

Geiger, Marsden, and Rutherford performed a scattering experiment with α particles from radioactive sources. By shooting α particles through thin gold foil targets, about $\sim \frac{1}{8000}$ were deflected through significant angles.

Example. Americium ^{241}Am, Radium ^{223}Ra, and Polonium ^{210}Po, where the number indicates the mass number (protons and neutrons) are radioactive. If ^{223}Ra produces $6MeV\,\alpha$ particles, are they relativistic if we use them in scattering experiments?

$$\gamma = 1 + \frac{K}{E_0} = 1 + \frac{6MeV}{(2m_p + 2m_n)c^2(eV)} = 1.0016 \approx 1.$$

$\Diamond$

The nucleus of a gold atom has a diameter of $d \approx 14 \cdot 10^{-15}m = 14fm$, which is about $\frac{1}{20000}$ the total diameter of the atom. While these are packed in a (polycrystalline) lattice and take up very little area, the α particles are much less massive than these nuclei.

- $m_e = 9.1094 \cdot 10^{-31}kg$
- $m_\alpha = 6.6447 \cdot 10^{-27}kg$, $\sim 7294m_e$
- $m_{Au} = 3.271 \cdot 10^{-25}kg$, $\sim 50m_\alpha$

Consider elastic scattering of α and e in the gold foil with a head-on collision for maximum momentum transfer.

$$m_\alpha v_\alpha = m_\alpha v'_\alpha + m_e v'_e.$$

However, due to the extreme discrepancy in masses $m_\alpha \gg m_e$, conservation of energy gives $v'_e \approx 2v_\alpha$. Therefore, the momentum of the α particle is virtually unchanged since $p_\alpha \gg p'_e$. If the α particle's momentum change is confined entirely to the perpendicular direction, the maximum deflection angle is (using small angle approximation):

$$\theta_{max} = \frac{|\Delta \boldsymbol{p}_{max}|}{|\boldsymbol{p}_\alpha|} \approx \frac{2m_e v_\alpha}{m_\alpha v_\alpha} \approx 2.74 \cdot 10^{-4} rad = 0.016^\circ.$$

Therefore, it is not possible that the scattering could be due to α-electron collisions.

Next, examine Coulombic scattering between the alpha particle and nucleus, subject to the impact parameter b. We assume that the nucleus is very massive and undergoes little recoil,

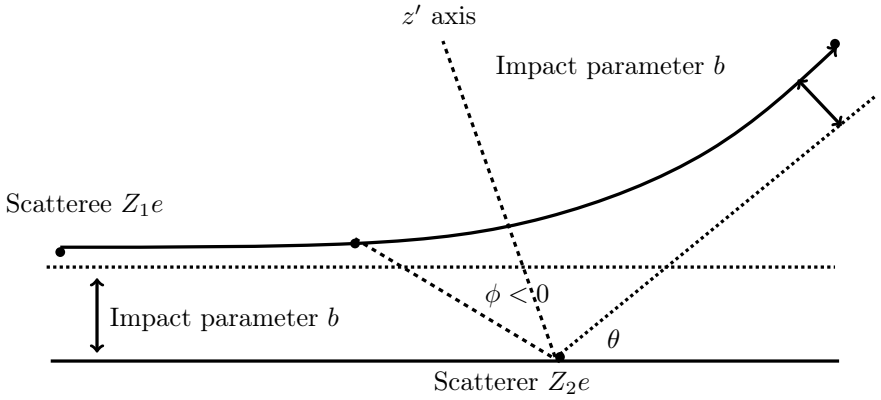


Figure 8: Rutherford Coulombic scattering between alpha and nucleus

while the target is very thin, so we consider only one scattering event. We will ignore non-Coulombic interactions, relativistic corrections, and rigid-body properties of the particles.

For a $6MeV\,\alpha$ particle, the minimum approach distance can be found with conservation of energy (impact parameter $b = 0$):

$$U = \frac{1}{4\pi\epsilon_0}\frac{Z_\alpha Z_{Au} e^2}{r_{min}} = K.$$

This gives $r_{min} \approx 3.79 \cdot 10^{-14} m$, which about five times greater than the radius of the gold nucleus, so they do not reach.

2026-01-30 **Lecture 12**

We know that the momentum is transferred along the z' axis, which is defined as $\boldsymbol{p}_f - \boldsymbol{p}_i$.

$$\Delta \boldsymbol{p} = \boldsymbol{p}_f - \boldsymbol{p}_i = 2mv_0 \sin\frac{\theta}{2}.$$

The only portion of the Coulomb force that contributes to the net change in momentum is $F\cos\phi$, where ϕ is the angle between the z' axis and the segment connecting the two particles. We can integrate:

$$\Delta p = 2mv_0 \sin\frac{\theta}{2} = \int F\cos\phi \, dt = \frac{1}{4\pi\epsilon_0} Z_1 Z_2 e^2 \int \frac{\cos\phi}{r^2} \, dt.$$

Since angular momentum is also conserved, the total angular momentum for $t \to -\infty$ applies $\forall t$:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = mv_0 r \sin\theta' = mv_0 b.$$

In a separate construction, as the particle moves a little bit along the path:

$$r d\theta' = dr \sin\theta'.$$

Taking the time derivative and multiplying by mr on both sides:

$$mr^2 \frac{d\theta'}{dt} = mv_0 b.$$

Which expresses r^2 in terms of constants and the θ' time derivative. Subbing back into our original integral, we can integrate over the particles entire path. Setting our two expressions for Δp equal and rearranging, assuming non-relativistic kinetic energy K:

$$b = \frac{Z_1 Z_2 e^2}{8\pi\epsilon_0 K} \cot\left(\frac{\theta}{2}\right).$$

This relates the impact parameter b and the scattering angle θ for a given K.

- For large b, θ is small
- If $\theta = 90°$, then b is small
- If $\theta = 180°$, then $b = 0$

Example. What is the impact parameter when scattering a $7.7MeV$ α particle from gold at and angle of $1°$ and $90°$?

For low angle, noting that the kinetic energy must be in Joules, and the prefix to the cot term is in m:

$$b = \frac{Z_1 Z_2 e^2}{8\pi\epsilon_0 K} \cot\frac{\theta}{2} = \frac{(2)(2)(e^2)}{8\pi\epsilon_0 eV} \cot\left(\frac{1°}{2}\right) = 1693fm.$$

For high angle:

$$b = 14.8fm.$$

Since $r = 7fm$, the impact parameter is about 2 radii of the Au atom. $\Diamond$

To obtain a probability, we use a *cross section* to classify scattering at angles greater than a set threshold angle θ.

$$\sigma = \pi b^2 \quad [m^2].$$

This is the area swept out by a set b around the target nucleus. Any $b_i \leq b_0$ will lie inside this cross section, and will produce scattering angles greater than θ_0.

We can calculate a probability using an area percentage. The number of cross sections per volume, $n = \frac{natoms}{unitvolume}$ can be multiplied by the volume of the foil $V = tA$. Taking an area ratio:

$$f = \frac{ntA \cdot \sigma}{A} = nt\sigma = nt\pi b^2.$$

Substituting in our value for b found earlier:

$$f = \pi nt \left(\frac{Z_1 Z_2 e^2}{8\pi\epsilon_0 K}\right)^2 \cot^2\left(\frac{\theta}{2}\right).$$

Give the fractional probability of a given scattering event.

2026-02-02 **Lecture 13**

Its important to note that the fraction of particle scattered as a specific θ is 0, since it is a continuous variable. However, we can measure the fraction of particles scattered through some $\theta \pm \epsilon$:

$$df = -\pi nt \left(\frac{Z_1 Z_2 e^2}{8\pi\epsilon_0 K}\right)^2 \cot\left(\frac{\theta}{2}\right) \csc^2\left(\frac{\theta}{2}\right) d\theta.$$

Note that the probability goes down as the angle goes up, hence the negative. We can now find the Rutherford scattering equation. The total number of particles scattered per unit area at $\theta \pm d\theta$ is $N_i\,(df)$:

$$N(\theta) = \frac{N_i df}{dA} = \frac{N_i nt}{16}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{Z_1^2 Z_2^2}{r^2 K^2} \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}.$$

As $\theta \to 0$, $N(\theta) \to \infty$, which actually saturates at the number of incident particles. As $\theta \to 180$, the count per area collapses to our prefactor.

The size of the nucleus

The Coulomb forces causes Rutherford scattering, measured centre to centre. If K_α is high enough to make 'contact' (within a few fm) with the nucleus, we will get deviations from Rutherford scattering predictions. At these distances, the strong forces kicks in, which creates a deep potential valley on the scale of the nucleons. We will first see these deviations at high angle scattering, since these incident particles' low impact parameter allows them to approach the nucleus most directly.

With non-relativistic conservation of energy assumptions:

- $K = 7.7MeV$ alpha particles upon Au: $r_{min} = 30fm$
- $K = 7.7MeV$ alpha particles upon Al: $r_{min} = 4.9fm$

2026-02-04 **Lecture 14**

2.2 Classical Atomic Model

After Rutherford scattering, the atomic picture emerged as electrons orbiting a small, planetary nucleus at the center of the atom. Let's use the model for some mathematical analysis.

First, assume the nucleus is stationary for our Hydrogen-like species, with an electron in orbit. Using Newton's Laws:

$$F = \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r^2} = \frac{mv^2}{r} \implies v^2 = \frac{Ze^2}{4\pi\epsilon_0 mr}.$$

We can get a dimensional ballpark estimate of $v = 0.008c$ for a electron in orbit around Hydrogen at $r \approx 0.5A$.

The total energy of the atom would then be:

$$E = K + V = 1/2mv^2 - \frac{Ze^2}{4\pi\epsilon_0 r} = -\frac{Ze^2}{8\pi\epsilon_0 r}.$$

Since the total energy is negative, it is a bound system. We can imagine it as sitting in the bottom of a potential well created by summing the electrical potential (attractive), and 'centrifugal potential' (repulsive), a fictitious potential representing the angular momentum. Observe that this definition sets $E = 0$ at $r = \infty$.

The magnitude of our E for a ground state represents the ionization energy of an electron at this radius, which comes out to about $14.4eV$ for Hydrogen. Comparing to the Rydberg formula for Hydrogen spectral lines:

$$\frac{1}{\lambda} = R_H\left(\frac{1}{n^2} - \frac{1}{k^2}\right).$$

And calculating the maximum possible photon energy emitted from Hydrogen (corresponding with a relaxation from $k = \infty \to n = 1$), we obtain $E = \frac{hc}{\lambda} = 13.6eV$, which is dimensionally about the same as our prediction!

Of course a fault with this model is that accelerating charge emits EMR, so electrons should be losing energy and collapsing into the nucleus. In response, the Bohr model is developed:

- Stationary states with definite energy do not radiate;
- Emission and absorption of radiation corresponds with exact transitions between energy states;
- Classical physics only describes dynamical equilibrium within a state, not transitions;
- Angular momentum of a stationary state is quantized:

$$L = mvr = \frac{nh}{2\pi} = n\hbar \implies v^2 = \frac{n^2\hbar^2}{m^2r^2}.$$

Combining our classical and quantized expressions for v^2, we can solve for r_n :

$$r_n = \frac{4\pi\epsilon_0 n^2\hbar^2}{mZe^2} = \frac{n^2}{Z}a_0.$$

Where $a_0 = \frac{4\pi\epsilon_0\hbar^2}{mZe^2} \approx 0.529A$ is the Bohr radius. Note that a_0 is defined specifically for Hydrogen.

The total energy then becomes:

$$E_n = -\frac{E_0}{n^2}, \quad E_0 = \frac{Z^2e^2}{8\pi\epsilon_0 a_0} = 13.6eV.$$

This describes the Hydrogen emission spectrum, but unlike the Rydberg formula, this is rooted in 'theory'. However, the prefactor $R_\infty \neq R_H$ between the Bohr and Ryder equations don't perfectly match, since we didn't account for the reduced mass correction our Bohr derivation. Alternatively, we get:

$$E_n = -\frac{\mu e^4}{8h^2\epsilon_0^2}\frac{Z^2}{n^2}.$$

For $\mu = \frac{m_1 m_2}{m_1+m_2}$ as the reduced mass correction. This stems from treating the angular momentum quantization as a single particle of mass μ orbiting at r_n.

2026-02-06

Lecture 15

Example. Find electron speed in ground state.

$$mv_n r_n = n\hbar \implies v_1 = \frac{\hbar}{ma_0} \approx 0.0073c.$$

An interesting result is the fine structure constant $\alpha = \frac{v_1}{c} = \frac{\hbar}{ma_0 c} = \frac{e^2}{4\pi\epsilon_0}\frac{1}{\hbar c} \approx \frac{1}{137.035999706}$. ◊

Example. Find the orbital period of a electron in the ground state.

$$T = \frac{2\pi a_0}{v_1} = 0.15fs.$$

◊

The reduced mass for our system is:

$$\mu_e = \frac{m_e M}{m_e + M} = \frac{m_e}{1 + \frac{m_e}{M}}.$$

We can correct our initial assumption that the proton is infinitely massive, leading to R_∞, by replacing that mass of the electron with the reduced mass in the Rydberg equation.

Generally, we can apply the Bohr model to any Hydrogen-like system, like positronium, He^+, Li^{++}, or other single-electron species. Note that the Bohr radius, or the smallest radius for a given Hydrogen-like species, $a_0 \propto \frac{1}{Z}$, which will impact our E_0.

2026-02-09

Lecture 16

Atoms also have characteristic x-ray emission profiles. When high energy electrons bombard a sample, they may 'hit' electrons in the $n = 1$ K shell, $n = 2$ L shell, and so on, which may ionize the electron. Electrons from upper shells will drop down to fill the vacancy, emitting a photon of x-ray energy. This follows the Rydberg formula:

$$\frac{1}{\lambda} = Z^2 R\left(\frac{1}{n_l^2} - \frac{1}{n_u^2}\right).$$

Where $Z = Z_{eff}$ is variable depending on the parameters of the transition. Nomenclature:

- An electron dropping one energy level into the K shell is associated with the K_α transition;
- Dropping two energy levels into the L shell is associated with the L_β transition, and so on.

The lower Z atoms are associated with lower frequency, or lower energy, K_α $n = 2 \to 1$ x-rays, since they hold electrons 'less tightly'. Moseley finds:

$$f_{K_\alpha} = \frac{3}{4} cR \left(Z - 1\right)^2 .$$

From the Rydberg formula $\frac{1}{\lambda_{K_\alpha}} = \left(Z - 1\right)^2 \left(\frac{1}{1} - \frac{1}{2^2}\right)$. Note that we use $Z - 1$ since the effective nuclear charge seen by an electron in the K shell is decremented by the other electron circulating in the K shell (2 electrons total).

We note that plotting elements against their K_α frequencies has a different slope that for their L_α frequencies. This is because the prefactor changes due to the differing integer energy levels involve din the transitions, and the different Z_{eff} seen by the transitioning electron.

Example. For example, the Z_{eff} for the L_α line is about $Z - 7.4$. This is due to the shielding of other electrons through this transition. ♢

In 1914, the Franck-Hertz experiment showed that atomic electron energy levels are quantized. They excited a filament in Hg vapour and accelerated the electrons across an electric field, measuring the current. They found that at increasing acceleration voltages, the collector current oscillated. They also sue a slight deceleration current to measure the electrons, and filter out low-energy electrons that have excited Hg atoms.

Above $4.88eV$ was the threshold energy to exciting Hg atoms, which would absorb this energy an release it as a photon. The oscillations are caused by how many times electrons are able to reach the minimum $4.88eV$ to excite Hg rather than being counted as current. Minima correspond with an integer number of $4.88eV$, leaving no remaining energy to get up the collector's reverse bias potential hill.

3 Wave Properties of Matter and Quantum Mechanics I

Crystal structures can act as perpendicular sets of lattices that can diffract x-rays since $\lambda \approx 0.06nm < d$. This is due to the spacing of planes causing path length differences upon reflection and interference. This gives Bragg's Law:

$$2d \sin\theta = n\lambda.$$

Where θ is the angle of incidence from the parallel with the planes, and d is the distance between planes. Note that the exit ray will be at 2θ from the incident ray due to law of reflection.

Lecture 17

2026-02-11

Recall that there are multiple families of planes, each of which has a different d and results in a peak on a Bragg peak diagram. If we use a polycrystalline material, we end up with sets of rings, each ring radius corresponds with a diffraction due to a specific plane and d, but forming a circle due to the varying orientations in the polycrystalline material.

3.1 De Broglie Waves

Inspired by the wave and particle nature of photons, he proposed that a particle with momentum p has an equivalent matter wavelength of:

$$\lambda = \frac{h}{p}.$$

Where this matter waves idea stems from photon momentum $E = hf = pc$. Note that this formula applies both relativistically and non-relativistically, depending on how p is calculated.

Example. Imagine an orbiting electron as a matter wave with De Broglie wavelength $\lambda = \frac{h}{p}$. A stationary state must have an integer number of wavelengths around a circle:

$$2\pi r = n\lambda = n\frac{h}{p} \implies L = rp = n\frac{h}{2\pi} = n\hbar.$$

We can use this relation to re-express kinetic energy in terms of quantized momenta, as well as the total energy of an electron matter wave. This is the same as our earlier method where we quantized angular momentum, but this time justifying this choice from the De Broglie perspective. ◊

In 1925, Davisson and Germer showed electrons having wavelike properties as they exhibited Bragg diffraction off of a Nickel single crystal. This principle is used in TEM with roughly $250keV$ electrons for super small wavelengths. This results in high resolution spectroscopy. Additionally, neutron scattering is used.

Example. 54 eV electrons are normal to a Nickel surface and a strong reflection is detected at only $\phi = 50°$. Using Bragg's Law, show that this implies an inter plane spacing of $D = 0.22nm$.

Note that $\phi = 50°$ with respect to the incident ray, which is a $\theta = 65$ degrees. Therefore:

$$d = \frac{n\lambda}{2\sin\theta} = 0.092nm.$$

This is the plane spacing along a diagonal plane of atoms at 25 degrees from the normal surface. A bit of trig shows $D = 0.22nm$. ◊

2026-02-13

Lecture 18

Example. A TEM has an acceleration voltage of $250kV$, what is the λ of the electrons?

We can find the wavelength with De Broglie's matter wave equation and relativistic momentum:

$$K = (\gamma - 1)\,mc^2 = 250keV \implies \gamma = 1 + \frac{250keV}{511keV} = 1.49 \implies u = 0.741c.$$

Finding momentum from relativistic kinetic energy:

$$p = \gamma mu = (1.49)\left(9.1094 \cdot 1^{-31}kg\right)\left(2.22 \cdot 10^8 \frac{m}{s}\right) = 3.02 \cdot 10^{-22}\frac{kg \cdot m}{s}.$$

Therefore, $\lambda = \frac{h}{p} = 0.022Angstroms <$ Bohr radii, so we can diffract through lattices. ◊

Example. Thermal neutrons from a nuclear reactor are at $T = 300K$. The thermal energy of a particle is:

$$K = \frac{3}{2}k_BT = \frac{3}{2}\left(1.381 \cdot 10^{-23}\frac{J}{K}\right)(300K) = 6.21 \cdot 10^{-21}J = 0.03888eV = 38.8meV.$$

Observe that $K < m_nc^2$ so we definitely don't need relativity!

$$K = \frac{p^2}{2m} \implies p = \sqrt{3mk_BT}.$$

Finding the wavelength:

$$\lambda = \frac{h}{p} = 1.45 Angstroms.$$

Which is small enough to diffract through gaps in crystal lattices. ◊

3.2 Wave Motion

The classical wave equation is:

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2}.$$

For waves on a rope with tension T and linear mass density μ, $v = \sqrt{\frac{T}{\mu}}$

When defining a possible sinusoidal wave travelling to the right, following the d'Alembert form:

$$\Psi(x,t) = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right] = A\sin\left[kx - \omega t\right].$$

We note that: $\lambda = vT = \frac{v}{f}$, $\omega = 2\pi f = \frac{2\pi}{T}$ are characteristic values. We can also introduce a phase constant ϕ that shifts the sinusoid:

$$\Psi(x,t) = A\sin\left[kx - \omega t + \phi\right].$$

With phase $kw - \omega t + \phi$. Notably, we have the wave number or wave vector, describing the periodicity in space:

$$k = \frac{2\pi}{\lambda}\left(m^{-1}, \frac{rad}{m}\right), \quad v_{phase} = \frac{\omega}{k}.$$

Where v_{phase} is the velocity of a point of constant phase. We can rewrite this as a dispersion relation for the wave:

$$\omega = v_{ph}k.$$

A linear combination of two solution to the wave equation also solves the wave equation, since it is a linear DE.

$$\begin{aligned}\Psi(x,t) &= A\cos\left[k_1 x - \omega_1 t\right] + A\cos\left[k_2 x - \omega_2 t\right] \\ &= 2A\cos\left[\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right]\cos\left[k_{avg}x - \omega_{avg}t\right].\end{aligned}$$

This is a product of a slowly oscillating amplitude envelope with a quickly oscillating pitch. The envelope moves with group velocity v_{gr}, while points of constant phase move with v_{ph}:

$$v_{gr} = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}, \quad v_{ph} = \frac{\omega_{av}}{k_{av}} = \lambda f.$$

Take Δx to be the length of a group packet, which it covers in Δt. Observe that:

$$v_{gr} = \frac{\Delta x}{\Delta t} = \frac{\Delta\omega}{\Delta k} \implies \begin{cases}\Delta k \Delta x = 2\pi \\ \Delta\omega\Delta t = 2\pi\end{cases}.$$

A Fourier series can be used to describe a periodic function, and the transform can be used to describe a single wave packet in space.

2026-02-23

Lecture 19

A periodic function can be described using a Fourier series:

$$\Psi(x,t) = \sum_i A_i \cos(k_i x - \omega_i t).$$

To construct a single wave packet, we can use a sinusoid with an envelope:

$$\Psi(x,0) = A\exp\left(-\Delta k^2 x^2\right)\cos(k_0 x).$$

For matter waves and photons, we have two characteristic equations:

$$E = hf = \hbar\omega$$
$$p = \frac{h}{\lambda} = \hbar k.$$

This leads to:

$$v_{gr} = \frac{dw}{dk} = \frac{dE}{dp} = \frac{pc^2}{E} = u.$$

Therefore, for matter waves, the group velocity of that wave packet is the actual velocity of the particle. This sort of intuitively makes sense, as the wave packet is the construction of our particle.

3.3 Wave Particle Duality

We have seen experiments demonstrate the particle and wave-like nature of both light and matter. Can we see both at the same time? Young's double slit experiment with light or electrons shows this via interference patterns (wave-like) affecting the particle impact distribution through double slits. Amazingly, this double-slit particle distribution appears even when sending one photon or electron at a time, implying self-interference.

Imagine using a beam of light to detect which slit the single photon or electron went through. To resolve the slits, we require:

$$\lambda_{ph} < d \implies p_{ph} = \frac{h}{\lambda_{ph}} > \frac{h}{d}.$$

However, the electrons or photons undergoing the double slit diffraction must obey a similar condition with respect to the slit separation, so $p_{ph} \sim p_{particle}$. This means the photon used to detect the electron will significantly impact the particle's momentum, and the interference pattern is lost. Now, we can get to Bohr's principle of complementarity, which is that we cannot describe observable phenomena (measured quantities like position, velocity, momentum, energy, etc.) in terms of both particles and waves at the same time.

3.4 Uncertainty Principle

For Gaussian wave packets, $\Delta k \Delta x = \frac{1}{2}$, but $k = \frac{2\pi}{\lambda} = \frac{p}{\hbar} \implies p = \hbar k$ Therefore:

$$\Delta p_x \Delta x \geq \frac{\hbar}{2}.$$

Which also follows for other coordinates by cyclic permutation.

Lecture 20

2026-02-27

Definition 9. The Heisenberg uncertainty principle in space is as:

$$\begin{cases} \Delta p_x \Delta x \geq \frac{\hbar}{2} \\ \Delta p_y \Delta y \geq \frac{\hbar}{2} \\ \Delta p_z \Delta z \geq \frac{\hbar}{2} \end{cases}.$$

Using $E = \hbar\omega$, in time this becomes:

$$\Delta E \Delta t \geq \frac{\hbar}{2}.$$

Let's first examine a classical oscillator with $k = \omega^2 m$. The total energy it possesses is:

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{(\Delta p)^2}{2m} + \frac{1}{2}m\omega^2 (\Delta x)^2 .$$

Note that we replace x and p by their uncertainties, rather than exact values, for a particle oscillating around equilibria of $x = p = 0$. Using our uncertainty relation, we can sub in Δp_{min} (for the ground state use minimum uncertainty) in terms of Δx and find the minimum energy by differentiating $\frac{dE}{d(\Delta x)} = 0$. This results in $E_{min} = \frac{\hbar\omega}{2}$, where ω is the resonance frequency of the oscillator. It is referred to as the quantum zero-point energy. This is the half-photon energy with frequency f.

To be able to observe this minimum energy, we must have $E_{min} > k_B T$, i.e. it must be greater than the thermal energy.

In defining matter in terms of waves, we end up with a minimum energy consequence for the example of a particle in a box.

Example. Imagine a particle confined a box of length L, and arbitrarily suppose $\Delta x = \frac{L}{2}$. Using the uncertainty principle, we obtain $\Delta p_{min} \approx \frac{\hbar}{L}$, which for a non-relativistic particle has kinetic energy:

$$K_{min} = \frac{p^2}{2m} \approx \frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{2mL^2}.$$

Note that Δp is like the standard deviation of momenta distribution in the box, where $\overline{p} = 0$. $\Diamond$

We can see that decreasing the size of the box increases the minimum kinetic energy it must possess, by consequence of the uncertainty principle on the matter wave and the de Broglie momentum relation.

Example. Confine an electron in a 10 Angstrom space. Using our relation above, the minimum energy the electron must possess is $0.038eV$, which is easily seen at lower temperatures as it is comparable to the thermal energy $\frac{3}{2}k_B T$. $\Diamond$

A consequence of this zero-point energy in terms of the particle in a box example is that Helium will never form a solid at 1 atm, because it's zero-point energy is greater than the binding energies of solidification. We can estimate it's zero-point energy in terms of the interatomic distance of $d \approx 3$ Angstroms.

$$E_{min} \approx \frac{\hbar^2}{2md^2}.$$

2026-03-02 **Lecture 21**

The vacuum is not empty! It is full of 'virtual' particles appearing and disappearing out of nothingness, on timescales 'too small for nature to realize'. They break the uncertainty principle:

$$\Delta E \Delta t < \frac{\hbar}{2}.$$

This includes creating of particle anti-particle pairs out of nothing, provided that $\Delta t_{annihilation} < \frac{\hbar}{2\Delta E}$.

Example. For an electron-positron vacuum fluctuation, the annihilation must occur in $t < 3 \cdot 10^{-22} s$ based on the rest energies involved. ◊

Vacuum fluctuations of EM radiation give rise to spontaneous emission of light from atoms. Vacuum fluctuations are the perturbations that cause the decay of higher energy states to lower energy states, such as relaxation of an excited atom leading to photon emission.

2026-03-04 **Lecture 22**

3.5 Bose-Einstein Condensate

As the temperature of a group of atoms decreases, the de Broglie wavelength as given by their thermal energy increases:

$$\lambda = \frac{h}{p}, p = \sqrt{2mK} \implies \lambda = \frac{h}{\sqrt{3mk_B T}}.$$

At high temperatures, the wavelength is much smaller than the interatomic distances $\lambda \leq d$. However, at very low temperatures, $\lambda \; d$, and the de Broglie wavelengths of particles connect to form a single macroscopic matter wave with a unified wave function. This is a Bose-Einstein Condensate.

Given the atomic density $n\left(\frac{\text{atoms}}{L^3}\right)$, the interatomic distances is $d \approx \frac{1}{n^{\frac{1}{3}}}$. Using approximate measures, we get:

$$n\lambda^3 = 1 \implies T = \frac{h^2 n^{\frac{2}{3}}}{3mk_B}.$$

As a rough condition for a BEC to form. Note that $n \sim 10^{14} cm^{-3}$ for dilute gas, or $n \; 10^{23} cm^{-3}$ in diamond. This predicts $T = 1.6 \mu K$ for Rb BEC, which is possible, and has been successfully done.

3.6 Wavefunctions

Matter waves are described by a wavefunction Ψ. Theh probability density of finding a particle within a certain region is given by $|\Psi|^2 = \Psi^* \Psi$. This means that we cannot assign physical meaning to Ψ, but only the probability density function. For a one-dimensional wavefunction:

$$\int_{-\infty}^{\infty} P(x)\, dx = \int_{-\infty}^{\infty} |\Psi(x,t)|^2\, dx = 1.$$

Measuring a particle in a given location results in the wave function instantaneously 'knowing' and collapsing at that point.

Note. The Copenhagen interpretation of quantum mechanics is based on:

- The Heisenberg uncertainty principles;
- Bohr's principle of complementarity (waves or particle at one given instance);
- Born's statistical interpretation of probabilities and the wave function.

△

Lecture 23

2026-03-06

3.7 Particle in a Box

Imagine a particle trapped in a $1D$ box of width L via an infinite square well potential. The wavefunction $\Psi = 0 \forall x < 0, x > L$. Using these boundary conditions $\Psi(0) = \Psi(L) = 0$, we find that the allowed standing waves satisfy:

$$n\frac{\lambda}{2} = L, n \in \mathbb{Z}^+ \implies \lambda_n = \frac{2L}{n}.$$

The energy of each of these standing waves is given by:

$$E_n = K_n + U = K_n = \frac{p_n^2}{2m} = \frac{h^2}{2m\lambda_n^2} = \frac{n^2h^2}{8mL^2}, n = 1, 2, 3, \ldots.$$

Note. Comparing to our uncertainty principle using $\Delta x = \frac{L}{2}$, we note that our estimate of the minimum energy was $E_{min} = \frac{h^2}{8\pi^2 mL^2}$. These two estimates differ by $\frac{1}{\pi^2}$. The Heisenberg uncertainty principle is based on a Gaussian that is not strictly limited by our infinite well boundary conditions, meaning there is a slightly larger wavelength, and therefore lower energy. △

Using our allowed λ_n, we get $k_n = \frac{2\pi}{\lambda_n} = n\frac{\pi}{L}$. Our wavefunction becomes:

$$\Psi_n(x) = A\sin(k_n x) = A\sin\left(n\frac{\pi}{L}x\right).$$

Given that the probability of finding the electron anywere in the box from $0 \to L$ is $P = 1$, we have a condition to find our amplitude:

$$\int_0^L A^2\frac{1}{2}\left(1 + \cos\left(2n\frac{\pi}{L}x\right)\right)dx = \frac{A^2}{2}L = 1 \implies A = \sqrt{\frac{2}{L}}.$$

Note. A smaller box corresponds with a higher base energy, and larger energy transitions. This results in short wavelengths being absorbed and emitted through excitation and relaxation events. △

4 Quantum Mechanics II

4.1 Schrodinger Wave Equation

We seek a wave equation governing matter waves in quantum mechanics that obeys conservation of energy, the de Broglie and Einstein relations, and linearity. Linearity must follow since Ψ is the motion of a particle and must be super imposable.

Take the ansatz $\Psi(x,t) = Ae^{i(kx-\omega t)}$, where A can be complex. We can simplify this as:

$$\Psi(x,t) = Ae^{ikx}e^{-i\omega t} = \psi(x)\,e^{-i\omega t}.$$

This allows us to cancel the time-dependence and use full derivatives with respect to x.

Note that from earlier, we can obtain:

$$k = \sqrt{\frac{2m(E-V)}{\hbar^2}}.$$

We can obtain the time-independent Schrodinger equation.

$$\frac{\partial^2\Psi}{\partial x^2} = -\frac{2m(E-V)}{\hbar^2}\Psi.$$

Definition 10. The time-independent Schrodinger equation is given by:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = E\Psi.$$

Definition 11. The full time-dependent Schrodinger equation is given by:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}.$$

Taking the magnitude of Ψ:

$$\begin{aligned}|\Psi(x,t)|^2 &= \Psi^*\Psi \\ &= \psi^*(x)\,e^{i\omega t}\psi(x)\,e^{-i\omega t} \\ &= |\psi(x)|^2.\end{aligned}$$

We can see that the time-dependence cancels naturally, making the probability density a *stationary state*.

Recall that the probability density given by $|\Psi(x,t)|^2$ must have unity area:

$$\int_{-\infty}^{\infty}\psi^*\psi\,dx = 1.$$

Meaning the wave function must be normalized. Ψ (or ψ) also have some important properties:

- Ψ must be finite;
- Ψ must be single-valued;
- For finite potentials V, Ψ and $\frac{\partial\Psi}{\partial x}$ must be continuous;
- Ψ must $\to 0$ as $x \to \pm\infty$.

Lecture 24

2026-03-09

Complex Numbers Review

We use the complex $i = \sqrt{-1}, i^2 = -1$. A generalized complex number $\tilde{z}$ has real and imaginary components, expressed in rectangular or polar form:

$$\tilde{z} = x + ij = Ae^{i\theta}, \quad A = |\tilde{z}|^2 = \tilde{z}\tilde{z}^* = \sqrt{x^2 + y^2}, \quad \theta = \arctan\frac{y}{x}.$$

We can draw a complex number on the s-plane as a phasor. Any phasor with $A = 1$ exists as a vector on the unit circle at an angle θ with respect to the x-axis. Therefore, $e^{i\frac{\pi}{2}} = i, e^{in2\pi} = 1$, etc.

Examining Euler's identity, $e^{i\theta} = \cos\theta + i\sin\theta$, we can obtain:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2j}.$$

As such, $e^{i(kx-\omega t)} = e^{ikx}e^{-i\omega t}$ can be broken down into a sum of sinusoids. Given that the Schrödinger equation is linear, sums of solutions to the differential equation are also solutions.

Recall that our coefficient $A = \tilde{A}$ can also be complex. We can write this in polar form to absorb the phase into our existing exponential, and leave a real coefficient.

4.1.1 Expectation Values

The expectation value of average x position given probability density $P(x)$ is a weighted average:

$$\langle x\rangle = \frac{\int_{-\infty}^{\infty} xP(x)\,dx}{\int_{-\infty}^{\infty} P(x)\,dx} = \int_{-\infty}^{\infty} xP(x)\,dx.$$

Where the wave function is assumed to be normalized.

Similarly, the expectation value of some observable Q is:

$$\langle Q\rangle = \int_{-\infty}^{\infty} \hat{Q}P(x)\,dx.$$

Where $\hat{Q}$ is the operator associated with observable Q.

Example. For example, the x-momentum operator is $\hat{p} = -i\hbar\frac{\partial}{\partial x}$. Operating on Ψ:

$$\hat{p}\Psi = -i\hbar\frac{\partial\Psi}{\partial x} = \hbar k\Psi = p\Psi.$$

$\Diamond$

Example. Let's solve a particle in a 1D square infinite potential well $x \in (0, L)$ with an infinite potential at the boundaries.

Take $\Psi(x,t) = \psi(x)e^{-i\omega t}$. Using the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \implies \frac{d^2\psi}{dx^2} = \frac{-2mE}{\hbar^2}\psi = -k^2\psi.$$

Where we are only considering $x \in (0, L)$ since the potential is 0 and can be dropped. By inspection, we can take:

$$\psi(x) = A \sin kx + B \cos kx.$$

With boundary conditions $\psi(0) = \psi(L) = 0$. Solving, we get:

$$\psi_n(x) = A \sin\left(\frac{n\pi}{L} x\right).$$

Normalizing, $A = \sqrt{\frac{2}{L}}$. To find the allowed energies, we can plug the wave function into the Schrödinger equation and the energy will pop out. Alternatively, we can use our known formula:

$$E_n = \frac{\hbar^2 k_n^2}{2m} = n^2 \frac{\hbar^2 \pi^2}{2mL^2}, \quad n = 1, 2, 3, \ldots.$$

$\Diamond$

Definition 12. *Correspondance Principle*: As $n \to \infty$ in quantum formulas, we approach the classical result.

Example. To find the probability of finding an electron trapped in an infinite square potential well in the left third of the box, we integrate:

$$P_{\frac{1}{3}} = \int_0^{\frac{L}{3}} P(x)\, dx = \frac{1}{3}\left[1 - \frac{3}{2}\frac{\sin\left(\frac{2n\pi}{3}\right)}{n\pi}\right].$$

As $n \to \infty$, we recover $P_{\frac{1}{3}} = \frac{1}{3}$, which is the classical result. $\Diamond$

We can represent a transition between states as a superposition of the individual wave functions of those states:

$$\Psi_{i \to j}(x, t) = \Psi_i(x, t) + \Psi_j(x, t).$$

Finding the probability density:

$$|\Psi_{i \to j}(x, t)|^2 = |\Psi_i|^2 + |\Psi_j|^2 + 2\Psi_i \Psi_j \cos\left(\frac{(E_j - E_i)\, t}{\hbar}\right).$$

Observe that this is the sum of each individual probability density plus a cross term that oscillates like a beat frequency. This is like the electron's probability density switching back and forth between states i and j as it transitions.

Note. This oscillation is like accelerating charges, which emits radiation! This can explain why photons are produced during transitions. $\triangle$

2026-03-11

Lecture 25

We have previously only considered infinite square well potentials, with which we can enforce the boundary conditions of $\psi(0) = \psi(L) = 0$. However, with a finite potential well:

$$V(x) = \begin{cases} 0 & 0 \le x \le L \\ V_0 & x < 0, x > L \end{cases}.$$

There exists a finite number of bound states, because above an energy of $E = V_0$, the particle is no longer bound. We will soon see that ψ will penetrate into the 'walls' of the box, but first let's solve the Schrödinger equation. Consider region 1, i.e. inside the box:

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} &= E\psi \\ \implies \psi(x) &= A\sin kx + B\cos kx. \end{aligned}$$

Now consider region 2, i.e. $x < 0$:

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}V_0\psi &= E\psi \\ \frac{\partial^2\psi}{\partial x^2} &= \frac{2m(V_0 - E)}{\hbar^2}\psi \\ \implies \psi(x) = Ce^{\alpha x} + De^{-\alpha x}, &\quad \text{defining } \alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar}. \end{aligned}$$

Similarly, in region 3, i.e. $x > L$:

$$\psi(x) = Fe^{\alpha x} + Ge^{-\alpha x}.$$

Note that the set $D = F = 0$, as the wave functions they produce are not bounded, and are unphysical. Therefore, we obtain a final:

$$\psi(x) = \begin{cases} Ce^{\alpha x} & x < 0 \\ A\sin kx + B\cos kx & 0 \le x \le L \\ Ge^{-\alpha x} & x > L \end{cases}.$$

Where ψ must be continuous and smooth, namely at our interval boundaries. The continuity condition demands:

$$C = B, \quad A\sin kL + B\cos kL = Ge^{-\alpha L}.$$

The smoothness condition demands

$$\alpha C = kA, \quad kA\cos kL - kB\sin kL = -\alpha Ge^{-\alpha L}.$$

Substituting and solving, we obtain the transcendental equation:

$$\frac{k}{\alpha} - \frac{\alpha}{k} = 2\cot kL, \quad \alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar}, \quad k = \frac{\sqrt{2mE}}{\hbar}.$$

Higher α leads to quicker decay, and eventually as $\alpha \to \infty$ collapses to our infinite well solutions.

$$\implies \frac{2E - V_0}{\sqrt{E(V_0 - E)}} = 2\cot\left(\frac{\sqrt{2mE}}{\hbar}L\right).$$

Which *must* be solved numerically in terms of E. Finding each of the intersections yields the finite number of E_n.

Note. The energy levels of a finite well for a given n are always smaller than those of the infinite well. This aligns with the uncertainty principle, which also yielded smaller energies than the infinite well. A more confined wavefunction has higher energies. △

When you are deep in the well, there is less penetration than at higher energies, since $\alpha \propto V_0 - E$. We define a penetration depth:

$$\delta = \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(V_0 - E)}}.$$

2026-03-13

Lecture 26

Now let's expand our perspective to $3D$ infinite potential wells. The wave function $\Psi(x,y,z) = \psi_1(x)\,\psi_2(y)\,\psi_3(z)$ now must obey the Schrödinger equation in all 3 dimensions.

$$\frac{p^2}{2m}\Psi + V\Psi = E\Psi.$$

Where the squared momentum decouples as:

$$p^2 = p_x^2 + p_y^2 + p_z^2.$$

Recall that $\hat{p_x}^2 = \hat{p_x}\hat{p_x} = \left(-i\hbar\frac{\partial}{\partial x}\right)^2$, so our total $\hat{p}^2$ is given by:

$$\hat{p}^2 = -\hbar\boldsymbol{\nabla}^2\Psi.$$

Where $\boldsymbol{\nabla}^2$ is the Laplacian of Ψ. Our $3D$ time-independent Schrödinger equation is then:

$$-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\Psi + V\Psi = E\Psi.$$

We finally obtain the net solution as the product of individual component solutions:

$$\Psi(x,y,z) = A\sin(k_1x)\sin(k_2y)\sin(k_3z).$$

Where infinite potential boundary conditions result in each $k_iL_i = n_i\pi$.

The energy levels for an electron in an infinite potential box are generally:

$$E = \frac{\pi^2\hbar^2}{2m}\left[\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2}\right].$$

This implies that the ground state energy is slightly larger, as each $n_1 = n_2 = n_3 = 1$.

For a cube, $L_1 = L_2 = L_3$, the symmetry of which simplifies analysis. The first excited states implies that one of $n_i = 2$ while the others remain $n_j = 1$. This results in 3 *degenerate*, or energy-equivalent first excited states due to the symmetry of our cube.

If we lift the degeneracy, or remove the symmetry in the system, we note that the energy level n_i along smaller dimensions (small L_i) have the largest impact on the net energy state of the system. The direction that is confined most is the biggest determiner of the overall energy.

4.2 Simple Harmonic Oscillator

Let's imagine a simple harmonic oscillator, which has a quadratic potential function of $V(x) = \frac{1}{2}m\omega^2x^2$ for $\omega = \sqrt{\frac{k}{m}}$. The classical turning points for this system are $x = \pm a$, where $V(\pm a) = E_{max}$. Subbing this potential into the time-independent Schrödinger equation:

$$\begin{aligned}\frac{d^2\psi(x)}{dx^2} &= \left(-\frac{2mE}{\hbar^2} + \frac{mkx^2}{\hbar^2}\right)\psi \\ &= \left(-\beta + \alpha^2x^2\right)\psi.\end{aligned}$$

The solutions to this differential equation are:

$$\psi_n(x) = H_n(x)\, e^{-\alpha \frac{x^2}{2}}.$$

Where H_n are called Hermite polynomial functions. The allowed energy levels are:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \quad n = 0, 1, 2, \ldots.$$

Note. We start our oscillator energy levels at $n = 0$, which is where the zero-point energy $E_0 = \frac{1}{2}\hbar\omega$ term comes from. △

Lecture 27

2026-03-16

Recall that in a finite well, a particle's wavefunction will penetrate into the potential 'walls' in increasing amounts as the particle is less deep in the well, or in higher energy states. Now imagine a situation where the potential function bounds a well with 'thin walls' (thickness $\sim \delta$). At higher allowed energy states, the probability density may penetrate enough into the potential barrier that that is non-zero ψ outside the potential walls. This leads to a possibility that the particle could end up *tunnelling* outside of the potential well.

There is an optical analogue to this: optical tunnelling. When total internal reflection occurs and photons perfectly reflect off of a surface from high to low n, the photon probability density penetrates slightly outside the surface. Putting another prism close enough to the first such that there is still an air gap, but the photon probability density penetrates into the second prism, results in *frustrated TIR*, where some light will be transferred.

A plane wave solution for a free particle is:

$$\Psi(x, t) = Ae^{ikx}e^{-i\omega t}.$$

Where e^{ikx} implies a right moving wave, while e^{-ikx} implies a left moving wave. Imagine a potential step is encountered, where the particle energy is greater than that of the step $E > V_0$. Treating the electron as a wave may result in both a reflected and transmitted portion of the solution. We can solve the Schrödinger equation for the two regions in our energy landscape.

$$\psi_{x<0} = Ae^{ikx} + Be^{-ikx}, \quad \psi_{x>0} = Ce^{ik'x}.$$

We have an incident component (A coefficient), a reflected component (B coefficient), and a transmitted component travelling in the same direction as the incident portion, with wave vector k'. To address our coefficients, we note that the wave function must be continuous and smooth at the boundary. Additionally, we must examine the particle flux:

$$\text{particles per unit time} = \frac{\Delta N}{\Delta t} = |\psi|^2 v = \text{prob. density} \times \text{velocity}.$$

Finding the transmission and reflection probabilities as:

$$T = \frac{C^*Ck'}{A^*Ak} = \frac{4kk'}{(k+k')^2}, \quad R = \frac{B^*B}{A^*A} = \left(\frac{k-k'}{k+k'}\right)^2.$$

Where $R + T = 1$. Since k and k' are related to E and V_0, we can express the reflection and transmission coefficients in terms of energy. If $E < V_0$, while the wavefunction may penetrate into the step, it cannot tunnel through the infinite potential step, so all of the energy will be reflected.

2026-03-18

Lecture 28

Example. Calculate the reflection probability for a $5eV$ electron encountering a step in which the potential drops by $2eV$.

Calculating the reflection coefficient:

$$R = \left(\frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E + \sqrt{E - V_0}}}\right)^2 = \left(\frac{\sqrt{5} - \sqrt{5 - -2}}{\sqrt{5 + \sqrt{5 - -2}}}\right)^2 = 0.00704 \approx 0.7\%.$$

Similarly, we can calculate the transmission probability and observe that $T + R = 1$. If the step went up from a $7eV \to 5eV$ difference, R and T would be unchanged. $\Diamond$

4.3 The Potential Barrier and Quantum Tunnelling

For particle with $E > V_0$ incident on a rectangular potential pulse, we can solve the Schrödinger equation for each of the three regions to find the reflected and transmitted radiation. We obtain:

$$R = \frac{\sin^2\left[\sqrt{2m\,(E - V_0)}\frac{L}{\hbar}\right]}{\sin^2\left[\sqrt{2m\,(E - V_0)}\frac{L}{\hbar}\right] + 4\frac{E}{V_0}\left(\frac{E}{V_0} - 1\right)}.$$

$$T = \frac{4\frac{E}{V_0}\left(\frac{E}{V_0} - 1\right)}{\sin^2\left[\sqrt{2m\,(E - V_0)}\frac{L}{\hbar}\right] + 4\frac{E}{V_0}\left(\frac{E}{V_0} - 1\right)}.$$

With $R + T = 1$ for the entire system. With k' found using the kinetic energy of the particles in the potential step:

$$K' = \frac{\hbar^2 k'^2}{2m} = E - V_0, \quad k' = \frac{2\pi}{\lambda'}.$$

And k similarly found but for the particles outside the potential step:

$$K = \frac{\hbar^2 k^2}{2m} = E, \quad k = \frac{2\pi}{\lambda}.$$

Now let's examine tunnelling, where $E < V_0$. We obtain:

$$R = \frac{\sinh^2\left[\sqrt{2m\,(V_0 - E)}\frac{L}{\hbar}\right]}{\sinh^2\left[\sqrt{2m\,(V_0 - E)}\frac{L}{\hbar}\right] + 4\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)}.$$

$$T = \frac{4\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)}{\sinh^2\left[\sqrt{2m\,(V_0 - E)}\frac{L}{\hbar}\right] + 4\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)}.$$

With $R + T = 1$ for the entire system. Imagine that $\alpha L = \frac{L}{\delta} \gg 1$, which is the *wide barrier approximation*. This demands:

$$\frac{\sqrt{2m\,(V_0 - E)}L}{\hbar} \gg 1.$$

Which occurs for an energy well below the potential step. In this case, there is a very low probability of tunnelling since our sinh term dominates the R and T terms, leading to $R \to 1$

and $T \to 0$. We can simplify our equation to find the tunnelling transmission probability in the wide barrier approximation as:

$$T \approx 16\left(\frac{E}{V_0}\right)\left(1-\frac{E}{V_0}\right)e^{-2\alpha L}.$$

Example. Tunnelling is relevant in scanning tunnelling microscopes (STM), field emission, alpha decay,... ◊

Lecture 29

2026-03-20

Electrons can be liberated from a metal in various ways. The first is thermionic emission, where electrons gain enough thermal energy to overcome the work function barrier. The second is field emissions, in which a strong external electric field generates a thin triangular potential barrier at the metal's surface. Electrons can then quantum tunnel through this barrier. As mentioned earlier, alpha decay is a similar tunnelling process, but where the potential triangle is generated by the radial limit of the strong force.

Let's examine some other examples of quantum tunnelling.

- Tunnel diodes

A high frequency oscillator with two quantum wells results in oscillation when the wells as at the same energy. When the levels are in alignment, the resonance condition results in conductivity.

- Ammonia molecule NH_3

NH_3 has a trigonal pyramidal structure with 1 lone pair on N. The nitrogen atom can tunnel through the hydrogen atoms to invert the triangle. This idea is the basis of MASERs, which in turn can be used as an atomic clock through the frequency of oscillation.

- Superconducting quantum Interference Devices (SQUID)

A tunnel current across a junction between two superconductors. These are qubit technologies, or a very sensitive magnetometer. It also forms the standard of the volt!

Example. Imagine a particle trapped between two thin potential barriers, with $E < V_0$. Is this a bound state?

Most textbooks will say no, due to the possibility of quantum tunnelling. While classically, it is a bound state, due to quantum effects, this might not last long! The particle will 'bounce' between the two walls at a certain 'attempt' frequency, with each collision having a probability of tunnelling. ◊

Example. What fraction of a beam of $50eV$ electrons would get through a $200V$, $1nm$ width electrostatic barrier?

Using $E = 50eV$, $V_0 = 200eV$, and $L = 1nm$ in our transmission coefficient formula:

$$T = \frac{4\left(\frac{E}{V_0}\right)\left(1-\frac{E}{V_0}\right)}{\sinh^2\left(\sqrt{2m\left(E-V_0\right)}\frac{L}{\hbar}\right)+4\left(\frac{E}{V_0}\right)\left(1-\frac{E}{V_0}\right)} = \frac{\frac{3}{4}}{\sinh^2\left(62.72\right)+\frac{3}{4}} = \frac{\frac{3}{4}}{7.5\cdot 10^{53}+\frac{3}{4}} = 9.95{\cdot}10^{-55}.$$

Note that $\alpha L = 62.72 \gg 1$, so we could have used our large-barrier approximation.

$$T \approx 16\left(\frac{E}{V_0}\right)\left(1-\frac{E}{V_0}\right)e^{-2\alpha L} = 9.98\cdot 10^{-55}.$$

If $E = 190eV$, redoing our calculations with the wide barrier formula obtains $T \approx 10^{-14}$.

Send a current of $1A = 1\frac{C}{s} = 6.24 \cdot 10^{18} e^-$ per second at the barrier. The elapsed time for a $150eVe^-$ to tunnel is:

$$\begin{aligned}\Delta t &= \frac{1}{\text{probability for one particle} \cdot \text{num particles per unit time}} \\ &= 1.6 \cdot 10^{35} s = 10^{27} \text{years}.\end{aligned}$$

For our $190eVe^-$, this is reduced to $\Delta t = 15\mu s$. ◊

Note. Using sinh, we must use **radians** and **radian mode**! △

2026-03-23

Lecture 30

Example. Consider a potential barier of $30eV$. Find the width around $1.000nm$ for which there will be no reflection of $35eV$ electrons incident upon the barrier. What would be the reflection probability for $36eV$ electrons upon this barrier?

We seek to find $L \approx 1nm$ such that the sin term in the numerator of R goes to 0. Using the fact that the zeroes of sin are $n\pi$, we have:

$$\sqrt{2m\left(E - V_0\right)}\frac{L}{\hbar} = n\pi.$$

Subbing in values, we find the index of the zero that we are looking for is $n = 3.64 \approx 4$. Resolving, we find $L = 1.097nm$.

Now, we can calculate the reflection probability for our $36eV$ electrons.

$$\begin{aligned}\alpha &= \sqrt{2m\left(E - V_0\right)}/\hbar = 1.254 \cdot 10^{10} m^{-1} \\ R &= \frac{\sin^2 \alpha L}{\sin^2 \alpha L + 4\left(\frac{E}{V_0}\right)\left(\frac{E}{V_0} - 1\right)} = 0.474.\end{aligned}$$

◊

Example. Let us model alpha decay as a rectangular potential barrier, starting at a distance of $r_{\text{nucleus}} = 7.4fm$ where the strong force disappears, and stretching to $r = r_{\text{nucleus}} \cdot \frac{35}{4.3}$. This is found by 'crushing' the true $V \propto \frac{1}{r}$ potential into a rectangular approximation, using $4.3MeV$ as our right bound due to the energy of incident α particles. The height of the rectangle is taken as $V_0 = \frac{1}{2}35MeV$, the max height of the true $\frac{1}{r}$ curve.

We can estimate the decay time for $U_{92}^{238} \to Th_{90}^{234}$ by finding the tunnelling probability as well as the attempt frequency.

$$\alpha = \frac{\sqrt{2m\left(V_0 - E = 17.5MeV - 4.3MeV\right)}}{\hbar} = 1.59 \cdot 10^{15} m^{-1} \implies \delta = 0.63fm.$$

Given that $L = \Delta r = 52.8fm$, the penetration depth is about 1% of this value, meaning our probabilty is going to be very small. Since $\alpha L = 83.8 \gg 1$, we can apply the wide barrier approximation:

$$T \approx 16\left(\frac{E}{V_0}\right)\left(1 - \frac{E}{V_0}\right)e^{-2\alpha L} = 3.57 \cdot 10^{-73}.$$

The attempt frequency can be found as $f = \frac{1}{\delta t}$ for round trip time $\delta t = \frac{v}{2r_{\text{nucleus}}}$. We can use the non-relativistic energy to find v, and then find the decays per unit time as:

$$fT = 3.47 \cdot 10^{-52} \implies \text{decay estimate of } 9.1 \cdot 10^{44}\text{years.}$$

$\Diamond$

Example. Let's continue our previous example, but this time working backwards from a known decay time of U^{238} of one decay per $6.5 \cdot 10^9$ years.

We will maintain our assumption of $4.3MeV$ α particles, implying that the transmission probability is:

$$T = 10^{-39}.$$

Again maintaining the wide barrier assumption, our new α can be found and verified against our assumption, which also leads to our required V_0:

$$T \approx e^{-2\alpha L} \implies V_0 = 7.95MeV.$$

$\Diamond$

Lecture 31 2026-03-25

5 The Hydrogen Atom

In the Hydrogen atom, the electron experiences a spherically symmetric, radial Coulombic force. However, the further the electron is confined, or pulled in, by the potential, the larger its minimum energy as per the Heisenberg Uncertainty Principle. This begs for spherical coordinates! The Laplacian in spherical coordinates is:

$$\boldsymbol{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial^2\phi}\right).$$

Similar to Cartesian, we can apply separation of variables to our wavefunction:

$$\psi(r,\theta,\phi) = \psi_r(r)\,\psi_\theta(\theta)\,\psi_\phi(\phi) = Rfg.$$

We substitute both of these into the time-independent Schrödinger equation.

$$\frac{fg}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{f}\frac{d}{d\theta}\left(\sin\theta\frac{df}{d\theta}\right) + \frac{1}{g}\frac{d^2g}{d\phi^2} + \frac{2\mu(E-V)}{\hbar^2}Rfg = 0.$$

Isolating the ϕ dependence, we can say that the ϕ-dependent side must be equal to some number, provided the other side is as well:

$$-m_l^2 = \frac{1}{g}\frac{d^2g}{d\phi^2}.$$

This has the solution:

$$g = e^{im_l\phi}.$$

Since $g(\phi) = g(\phi + 2\pi)$ by the periodicity of spherical polar coordinates, we obtain:

$$e^{im_l 2\pi} = 1 \implies m_l = 0, \pm1, \pm2, \ldots.$$

Following our earlier assumption, the right side of the equation must also equal $-m_l^2$. Again, we isolate each of our variable dependencies on either side of the equation, and set both sides equal to the constant $l\,(l+1)$.

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2\mu r^2}{\hbar^2}\,(E-V)=\frac{m_l^2}{\sin^2\theta}-\frac{1}{f\sin\theta}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{df}{d\theta}\right)=l\,(l+1)\,.$$

We try the ansatz $R=Ae^{-\frac{r}{a_0}}$ for the radial equation obtaining:

$$\left(\frac{1}{a_0^2}+\frac{2\mu E}{\hbar^2}\right)+\left(\frac{2\mu e^2}{\hbar^2 4\pi\epsilon_0}-\frac{2}{a_0}\right)\frac{1}{r}=0.$$

Which implies that each bracketed term must identically be 0 for the equation to hold $\forall r$. The first yields that $E_0=-13.6eV$, which is consistent with the Bohr model. The second yields the same expression for the Bohr radius (with the reduced mass) as the Bohr model.

The solutions to the angular and azimuthal equations yields spherical harmonics:

$$Y\,(\theta,\phi)=f\,(\theta)\,g\,(\phi)=Y_{l,m_l}.$$

Therefore, the net wavefunction is:

$$\Psi_{nlm_l}\,(r,\theta,\phi)=R_{nl}\,(r)\,Y_{lm_l}\,(\theta,\phi)\,.$$

Note. All $l=0$ states are spherically symmetric, but this changes for $l\geq 1$. The l number is associated with the *spdf* orbitals. △

Definition 13. The principal quantum number $n=1,2,3,4,\ldots$ describes the energy level and results from the radial component of the Schrödinger equation.

The orbital angular momentum quantum number $l=0,1,2,\ldots,n-1$ quantizes the angular momentum.

The magnetic quantum number $m_l=-l,-l+1,\ldots,-1,0,1,\ldots l-1,l$ tells us about the magnetic moment.

The orbital angular momentum quantum number describe:

$$\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{p}\implies|L|=mv_{\text{orbital}}r=\sqrt{l\,(l+1)}\hbar.$$

Large $|L|$ corresponds with less-elliptical orbitals, so $l=0$ corresponds with 0 angular momentum and therefore a trajectory through the nucleus!

2026-03-27

Lecture 32

The orbital angular momentum quantum number describes the quantized angular momentum in our orbitals:

$$\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{p},\quad L=\sqrt{l\,(l+1)}\hbar.$$

The idea is that the higher the l number, the more circular the orbit becomes. For $l=0$, or an s orbital, the angular momentum of 0 corresponds with an electron's orbit that passes through

the nucleus. This disagrees with our Bohr model of $L = n\hbar$, where all angular momentum dependence is contained in the energy level.

Note that there is azimuthal dependence ϕ of the z-component of the angular momentum:

$$L_z = m_l \hbar.$$

Since L_z is quantized, then the vector $\boldsymbol{L}$ can only point in certain directions, or exhibits *space quantization.*

Example. For $l = 2$, $m_l = \pm 2, \pm 1, 0$, we have 5 possible orientations for $\boldsymbol{L}$. Note that $L = \sqrt{2\,(2+1)}\hbar = \sqrt{6}\hbar$. For $m_l = 0$, the $\boldsymbol{L}$ is confined to the xy-plane. For $m_l = -2$, $L_z = -2\hbar$ and $L_{xy} = \sqrt{2}\hbar$. In the xy plane, $\boldsymbol{L}$ can precess around the z-axis. Note that $\boldsymbol{L}$ can never point directly along the $\pm z$-axis. ◊

Note. If $\boldsymbol{L}$ were to point along the z-axis, it would violate the uncertainty principle. This would imply that for $z = 0$, we would get $p_z = 0$, which does not obey:

$$z p_z \geq \frac{\hbar}{2}.$$

△

These properties mean that an external magnetic field will split spectral lines. Each electron orbiting around the nucleus has magnetic moment of:

$$\boldsymbol{\mu} = -\frac{e}{2m}\boldsymbol{L}.$$

In an external magentic field, the dipole will seek to align with the field to minimize magenetic potential energy:

$$U_\beta = -\boldsymbol{\mu} \cdot \boldsymbol{\beta}.$$

Resulting in a torque on the dipole of:

$$\tau = \boldsymbol{\mu} \times \boldsymbol{\beta}.$$

This torque causes a precession of $\boldsymbol{\mu}$ about $\boldsymbol{\beta}$ at the *Larmor precession frequency* ω_L.

Examining the z-component of the magnetic moment:

$$\mu_z = -\frac{e}{2m}L_z = -\frac{e}{2m}m_l\hbar = -\mu_B m_l.$$

Where $\mu_B = \frac{e\hbar}{2m_e}\left(\frac{eV}{T}\right)$ is the Bohr magneton, the fundamental quantized unit of magnetic moments. Therefore, our net potential energy is:

$$U_B = -\mu_z \beta = \mu_B m_l \beta.$$

This results is a slight difference in energy between the orbitals with differing m_l but otherwise identical n and l quantum numbers. As such, the emitted line spectra will have line splitting due to the differing magnetic moment, and therefore differing magnetic potential energies, involved in the transition. However, this only occurs in the presence of an external magnetic field!

Example. The size of splitting due to the Zeeman effect is very small. For $E_{211} \to E_{100}$ versus $E_{210} \to E_{100}$ in a field of $1T$:

$$\Delta E = \mu_B B = 5.79 \cdot 10^{-5}\frac{eV}{T}.$$

Which is difficult to see! ◊

2026-03-30 **Lecture 33**

5.1 Spin

The Stern-Gerlach Experiment

A uniform magnetic field will produce a net torque, but not force, on objects. We can use a non-uniform magnetic field to create a net force on atoms passing through a region to show a difference in m_l. Due to magnetic dipole moments in different directions, this will result in a splitting of atoms as they pass through to a screen on the far side.

This process was completed in the Stern-Gerlach experiment, and observed a splitting of atoms into two lines under the influence of a field, but no spitting without a field. They used Hydrogen with $l = 1$ and $m_l = 0, \pm 1$, and would have expected three lines. Additionally, when they used Hydrogen with $l = 0$, where there should be no differing magnetic moments due to $m_l = 0$, they still saw splitting!

This was contrary to what they expected, but led to the discovery of a new quantum number. They determined that the 'spin' on the electron cloud results in an intrinsic angular momentum and therefore magnetic moment. This isn't actually spinning though, it's a purely quantum mechanical effect.

Definition 14. The intrinsic spin quantum number s describe the intrinsic angular momentum of a particle ($s = \frac{1}{2}$ for e^-). The magnetic spin quantum number m_s ($m_s = \pm\frac{1}{2}$ for e^-) describes the projection of the intrinsic spin.

The spin component of the angular momentum is:

$$|\boldsymbol{S}| = \sqrt{s\,(s+1)}\hbar \implies S_z = m_s\hbar = \pm\frac{1}{2}\hbar.$$

Therefore, the intrinsic magnetic moment of an electron is given by:

$$\mu_{sz} = 2m_s\mu_B.$$

Protons also have intrinsic spin of $s = \frac{1}{2}$. more generally, we define *fermions* to be particles with half-integer spin. *Bosons* as particles with integer spin.

Definition 15. Pauli's exclusion principle (for fermions) states that no two electrons in an atom my have the same set of four quantum numbers.

If we could the number of quantum number arrangements for a given energy level n, we result in alignment with the periodic table.

2026-04-01 **Lecture 34**

5.2 Fine Structure

A magnetic field from the orbital angular momentum of an electron interacts with the spin of the electron, for $l > 0$. In the electron frame, the proton orbiting results in a magnetic field that

can interact with the spin up or spin down electron. We can estimate the magnitude of this field. From the Bohr model, in $n = 2$, the radius of orbit is $r = 4a_0$. The kinetic energy is:

$$\frac{1}{2}mv^2 = E_2 - V = -\frac{E_0}{4} + \frac{e^2}{4\pi\epsilon r} = 3.41eV \implies v = 1.10 \cdot 10^6 \frac{m}{s}.$$

The effective current from our orbit is:

$$I = \frac{dq}{dt} = e\frac{v}{2\pi r}.$$

Therefore, the field at the centre of the loop is:

$$B = \frac{\mu_0 I}{2\pi r} \approx 0.4T.$$

This results in a self-induced splitting of a $2p$, or other $l > 0$ orbital, into two non-degenerate states differing in spin. One spin electron will increase in energy by $\mu_B B$, while the other spin electron decreases by $\mu_B B$. This *doublet* will have an energy splitting of $\Delta E \approx 0.8\mu_B = 4.5 \cdot 10^{-5} eV$.

Example. An example of spin-orbit coupling (fine structure) is the sodium yellow doublet.

Sodium is $[Na] = 1s^2 2s^2 2p^6 3s^1$, and the transition forming this doublet is the $3p \to 3s$ transition. The $3p$ orbital experiences this spin-orbit coupling, resulting in the D_1 line $\lambda_1 = 589.6nm$ and the D_2 line $\lambda_2 = 589.0nm$. Note that λ_2 is higher energy, and therefore originates from the orbital that has increased in energy by $\mu_B B$. We can work backwards to find the magnetic field inducing this splitting, using $\Delta E = 2.14meV = 2\mu_B B$. The effective field is about $B = 18T$. $\Diamond$

5.3 Hyperfine Structure

In the ground state of hydrogen, $1s$, there is no orbital angular momentum since $l = 0$. However, both the electron and proton are fermions, meaning half-integer spin, and therefore have magnetic moments. Even with no orbital angular momentum, we can still have the magnetic moments of the individual spins interacting with each other.

When the dipoles are aligned, it's a higher energy state, and a lower energy state when they are anti-aligned. This causes splitting on the order of $10^{-6} eV$. Transitioning between these two states results in a $\lambda = 21.1cm$ photon. This property can map the distribution of hydrogen in our universe.

5.4 Transition Selection Rules

We note that 'allowed transitions' have Δn as anything, but restricts $\Delta l = \pm 1$. This means that transitions from adjacent l number orbitals is probabilistically favoured. 'Forbidden transitions' are anything with $\Delta l \neq \pm 1$, for example transitioning between s orbitals, or between p orbitals, or from s to d, etc. These transitions are lower probability.

Recall that transitions between energy levels is an oscillation of superimposed wave functions. They form an antenna-like structure. This is what generates the emitted radiation. However, when transitioning between the $1s$ and $2s$ states, for example, we don't get the same 'straight antenna', as there is no dipole moment between the two wave functions. It can happen, but it is less efficient. They happen over longer times, so by uncertainty principles:

$$\Delta E \Delta t \geq \frac{\hbar}{2}.$$

The lower probability has a longer Δt, resulting in a much narrower line width.

2026-04-08 **Lecture 35**

Examining the radial probability differential element in spherical coordinates:

$$P(r)\,dr = r^2 R^* R dr \int_0^\pi |f(\theta)|^2 \sin\theta\, d\theta \int_0^{2\pi} |g(\phi)|^2\, d\phi.$$

Note that we take advantage of our separation of variables to split our integrals. We obtain:

$$P_{nl}(r) = r^2 |R_{nl}(r)|^2.$$

Example. Calculate the radial probability density for an electron in the $1s$ state. What is the most probable distance from the nucleus?

We have $n = 1$ and $l = 0$, getting $R_{10}\frac{2}{a_0^{\frac{3}{2}}}e^{-\frac{r}{a_0}}$.

$$P_{nl}(r) = r^2|R_{nl}(r)|^2 = r^2\frac{4}{a_0^3}e^{-\frac{2r}{a_0}}.$$

We can the derivative to find the 0 of the probability density function, finding a critical point at $r = a_0$. The maximum probability density occurs at a_0, although the expectation value may be shifted. ◊

6 Special Topics

6.1 Atomic Physics

Definition 16. The Pauli Exclusion Principle states that no two electrons in an atom may have the same set of quantum numbers n, l, m_l, m_s.

This principle applies to all fermions (half-integer spins).

In the periodic table, we denote the electron shells based on the principle quantum number $n = 1, 2, 3, \ldots$, which we also label the $K, L, M, \ldots$ shell. Electron sub shells correspond with the combination of n and l, like $1s, 2s, 2p, 3s, \ldots$. Since each l value has $2l + 1$ m_l values, this describes the number of orbitals in each subshell. Therefore, the $l = 0$ s subshell has 1 orbital, the $l = 1$ p subshell has 3 orbitals, etc. The electronic configuration is how we fill up these shells and subshells.

Let's examine the configuration of Na.

$$[Na], Z = 11 : 1s^2 2s^2 2p^6 3s^1 \implies [Ne]3s^1.$$

When we looked at the Sodium doublet, the $3s^1$ electron moves between different $3p$ orbitals. But in these orbitals, why are the s orbitals lower energy than the p orbitals? In hydrogen, all subshells of the same n are degenerate. In multielectron atoms, the outer electron shells experience shielding, which makes the s orbital lower in the potential well than the p orbital. Similarly, the $4s$ orbital is lower energy than the $3d$ subshell, and therefore is filled first, due to penetration and effective nuclear charge.

6.2 Statistical Physics

Why does Pauli exclusion arise, and only apply to fermions, but not bosons? Consider a wave function that describes a system of two identical particles $\Psi(1,2)$. The way we describe the symmetry of this system will change, depending on if they are fermions or bosons. If we interchange the two particles, we should obtain the exact same probability density:

$$|\Psi(1,2)|^2 = |\Psi(2,1)|^2.$$

Therefore $\Psi(1,2) = \pm\Psi(2,1)$. The probability density is not affected, but the wavefunction has two possibilities. Bosons have symmetric wave functions:

$$\Psi(1,2) = \Psi(2,1).$$

While fermions have anti-symmetric wave functions:

$$\Psi(1,2) = -\Psi(2,1).$$

If we define a symmetric wavefunction:

$$\Psi_S(1,2) = \frac{1}{\sqrt{2}}\left[\psi_a(1)\psi_b(2) + \psi_a(2)\psi_b(1)\right].$$

Then the antisymmetric version is:

$$\Psi_A(1,2) = \frac{1}{\sqrt{2}}\left[\psi_a(1)\psi_b(2) - \psi_a(2)\psi_b(1)\right].$$

Fermions demand this anti-symmetric version. But what if the two fermions want to occupy the same state?

$$\Psi_A(1,2) = \frac{1}{\sqrt{2}}\left[\psi_a(1)\psi_a(2) - \psi_a(2)\psi_a(1)\right] = 0.$$

This function is now symmetric, and therefore breaks our symmetric fermion wavefunction principle.

Lecture 36

2026-04-10

6.3 Fermi-Dirac Statistics

What happens if we put many (non-interacting) particles in a box? For Fermions, we have to use Pauli exclusion, meaning each additional electron must either enter an empty energy level with any spin, a half-filled level with the opposite spin, or the next unoccupied energy level. The highest occupied energy is called the *Fermi energy*.

The probability that an electron occupies a state at energy E at a given temperature T is given by the *Fermi-Dirac distribution*.

$$f_{FD} = \frac{1}{\exp\left(\frac{E-E_f}{k_B T}\right) + 1}.$$

At a temperature of 0, the function is basically a step-down at the Fermi energy E_f, changing from 1 for $E < E_f$ to 0 for $E > E_f$.

At a finite temperature, the curve rounds out at the corners, or 'thermal smearing' of the distribution. The electrons closest to the Fermi energy have the chance to be excited by thermal fluctuations into energies above E_f. However, for $E - E_f \gg k_B T$, the distribution is dominated by the exponential term $\exp\left(\frac{E-E_f}{k_B T}\right)$.

How does a metal's Fermi energy interact with the photoelectric effect?

$$K = hf - \phi.$$

For $T = 0$, our Fermi-Dirac distribution shows that $\phi = E_{vac} - E_f$ will be very well defined (where E_{vac} is the vacuum energy), as above E_f, there is 0 probability of finding electrons. For $T \neq 0$, we have a range of energies. As such, ϕ will vary depending on E of the electrons, affecting the output kinetic energy K_{max} imparted on photoelectrons.

The smearing of the Fermi-Dirac distribution also implies that if we increase the temperature enough, the smearing will allow some electrons to have energies greater than the ionization energy gap E_{vac}. This is thermionic emission, where we effectively boil electrons off of the metal.

For bosons, which do not obey Pauli exclusion principle, there is no limit to how many bosons can occupy a given state. This is what allows for Bose-Einstein condensates.

A Bose-Einstein condensate is where the wave function will unify and span the entire material. Recall that ^4He is a boson as the net spin of the nucleons is 0, and the electron spins cancel to 0. As such, condensing ^4He to very low temperatures results in the fluid losing all viscosity as there is no resistance to the flow of atoms.

6.4 Bose-Einstein Statistics

Photons are bosons with spin 1. The Bose-Einstein occupancy is:

$$f_{BE} = \frac{1}{\exp\left(\frac{hf}{k_B T}\right) - 1}.$$

Which matches part of our distribution of the intensity of a blackbody at a given λ and T:

$$I(\lambda, T) = \frac{2\pi c^2 h}{\lambda^5} \frac{1}{\exp\left(\frac{hf}{k_B T}\right) - 1}.$$

The prefactor is our classical statistical physics prediction of the number of modes at a given wavelength. The average number of photons excited in each mode (resonant frequency) at some temperature T is:

$$\overline{n} = \frac{1}{\exp\left(\frac{hf}{k_B T}\right) - 1} = \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}.$$

We had found that the average energy was given by:

$$\overline{E} = \overline{n} hf \implies u(\lambda) = N(\lambda)\,\overline{E}(\lambda).$$

The density of modes at a given wavelength is $\propto \frac{1}{\lambda^5}$, while the energy of modes at a given wavelength increases roughly linearly. Their product gives the blackbody curve!

6.5 Superconductivity

Superconductivity is a macro-scale quantum phenomenon. Electrons form Cooper pairs with twice the electron mass and charge. They have a single wavefunction:

$$\Psi(\boldsymbol{r}) = \sqrt{n_{cp}} e^{i\theta \boldsymbol{r}} = \sqrt{n_{cp}} e^{i\boldsymbol{p}/\hbar \boldsymbol{r}}.$$

Where n_{cp} is the Cooper pair density.

Applying the momentum operator:

$$\hat{p}\psi = -i\hbar \frac{d}{dx}\psi = \hbar k \psi = p\psi.$$

The two main properties are 0 resistance and the Meissner effect, i.e. rejection of all magnetic fields. It does so spontaneously, generating currents to counter the magnetic fields inside of it in its normal state.

Lecture 99: Intro to Relativity 2026

7 Relativity

7.1 Introduction

Imagine a frame of reference S' moves with velocity $v\hat{x}$ with respect to an inertial frame of reference S. In order to describe an event happening at $(x, y, z, t)_S$ in S', we must transform our coordinates.

The canonical Galilean transformations are as follows:

$$\begin{aligned} x &= x' + vt \\ y &= y' \\ z &= z' \\ t &= t'. \end{aligned}$$

This assumes that both frames make the same time measurements. As such, differentiating with respect to $dt = dt'$, we get our velocity and acceleration relations. However, this does not follow Einstein's postulates, since a beam of light travelling along the x-axis at speed c in S travels at an observed speed $c - v$ in S'.

In reality, a spherical pulse emanating from the origin should have radius $r = ct$, and $r' = ct'$ in the two frames, since they both must observe the same value of c. In S, transforming this relation to Cartesian coordinates:

$$r = ct \implies \sqrt{x^2 + y^2 + z^2} = ct.$$

Therefore, in our two frames, we have:

$$\begin{aligned} 0 &= x^2 + y^2 + z^2 - c^2t^2 \\ 0 &= x'^2 + y'^2 + z'^2 - c^2t'^2. \end{aligned}$$

Since S' moves relative to S only along $\hat{x}$, then both $y = y'$ and $z = z'$, and therefore our relation collapses to:

$$x^2 - c^2t^2 = x'^2 - c^2t'^2.$$

Which cannot be satisfied by the Galilean transformations.

7.2 Derivation

We know that our desired transformation must be linear for a few reasons:

- Spacetime is considered to be homogeneous, i.e. there is no special 'origin' point. However, non-linear relations (e.g. x^2) would violate this principle as the choice of origin would drastically change the results. Linearity ensures that transformations are affine (meaning linear + possible translations), and simply defining that our frames coincide at $x = 0, t = 0$ results in these constants falling away.
- We know that the Lorentz transform should have a well defined inverse, in the sense that looking at S from S' should follow a conserved form to when looking at S' from S. Non-linear relationships would be directional, again breaking homogeneity.
- Take a new frame S'' that is to S' as S' is to S. The process of transforming from $S \to S'$ should be identical to that from $S' \to S''$. This means that 'boosts,' or changes in velocity, should form a group, while non-linear functions often are not closed under composition or inversion in this simple way.

Hopefully these loose conceptual arguments can convince you enough to accept the ansatz:

$$x' = Ax + Bt, \quad t' = Cx + Dt \tag{1}$$

Note. A side note: the determinant of the matrix form of our transformation $AD - BC$ will happen to equal 1. The determinant $\neq 0$ by the simple fact that the transformation must have an inverse. But the fact that it is equal to 1, and therefore area preserving in 1+1 dimensions, is pretty neat. △

First, our imposed condition $x = x' = 0$ at $t = t' = 0$ is automatically satisfied by our assumed linear relationship in Equation 1. Our arbitrary definition of zeroed constants allows us to avoid the more generally true affine relationship (which includes translation by a constant). We also know that S' moves at speed v in S, meaning that a value $x = vt$ should always map to $x' = 0$:

$$A(vt) + Bt = 0 \implies B = -Av \implies x' = A(x - vt) \tag{2}$$

From the forwards transformation:

$$x' = Ax - Avt$$
$$t' = Cx + Dt.$$

We can create a matrix form Λ for $S \to S'$:

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} A & -Av \\ C & D \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = \Lambda \begin{bmatrix} x \\ t \end{bmatrix}.$$

Where the matrix inverse should correspond with the transformation from $S' \to S$ (since it is a linear operator):

$$\begin{bmatrix} x \\ t \end{bmatrix} = \Lambda^{-1} \begin{bmatrix} x' \\ t' \end{bmatrix} = \frac{1}{AD + ACv} \begin{bmatrix} D & Av \\ -C & A \end{bmatrix} \begin{bmatrix} x' \\ t' \end{bmatrix}.$$

Examining our x equation:

$$x = \frac{Dx' + Avt'}{AD + ACv}.$$

Recall that we previously used the fact that at any time t in S, the coordinates of the origin of S' are at $x = vt$. Therefore, $x = vt$ should always map to $x' = 0$ in our transformation. We can

now use symmetry to imply the reverse reasoning. The origin of S moves at $-v\hat{x}$ when viewed from the S' frame, so $x' = -vt'$ should always map to $x = 0$:

$$0 = \frac{-Dvt' + Avt'}{AD + ACv} \implies A = D.$$

Our transformation is now:

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} A & -Av \\ C & A \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{3}$$

Now, we must one of Einstein's postulates:

- The invariance of the speed of light, $x = ct \implies x' = ct'$;
- The conservation of the spacetime interval $x^2 - c^2t^2 = x'^2 - c^2t'^2$

Subbing the first into our transformation in Equation 3:

$$\begin{bmatrix} ct' \\ t' \end{bmatrix} = \begin{bmatrix} A & -Av \\ C & A \end{bmatrix} \begin{bmatrix} ct \\ t \end{bmatrix}.$$

Algebra yields the relation $C = -\frac{v}{c^2}A$

For consistency of the forwards and reverse transformations, our matrix must satisfy the identity $\Lambda(v)\Lambda(-v) = \mathbb{I}$:

$$\begin{bmatrix} A & -Av \\ -\frac{v}{c^2}A & A \end{bmatrix} \begin{bmatrix} A & Av \\ \frac{v}{c^2}A & A \end{bmatrix} = \mathbb{I}$$

$$A^2 \begin{bmatrix} 1 - \frac{v^2}{c^2} & 0 \\ 0 & 1 - \frac{v^2}{c^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\implies A = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

This scenario corresponds with transforming from $S \to S'$ and then back again; we should end up with our original result, sans transformation.

By convention, we define $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, and can now write our final transformations as:

$$x' = \gamma(x - vt)$$
$$t' = \gamma\left(t - \frac{xv}{c^2}\right).$$

Note. There are clever geometric derivations of the Lorentz transformations that use a scenario like light bouncing of a mirror in a train, alongside the frame-invariance of c. △

Let's do a little more mathematical manipulation of the matrix form of our transformation, which we will be able to give graphical meaning to later. It is convenient to reformulate the transformation to be nice and symmetric:

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \gamma \begin{bmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \to \begin{bmatrix} x' \\ ct' \end{bmatrix} = \gamma \begin{bmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{bmatrix} \begin{bmatrix} x \\ ct \end{bmatrix}.$$

We achieved this by dealing in a scaled ct and ct'. We can define a new dimensionless variable called the rapidity α:

$$\tanh\alpha = \frac{v}{c}.$$

And then use hyberbolic trig identities to obtain:

$$\begin{bmatrix} x' \\ ct' \end{bmatrix} = \begin{bmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{bmatrix} \begin{bmatrix} x \\ ct \end{bmatrix}.$$

This is a hyperbolic rotation through angle α. Compared to a normal rotation, in which points slide around a circle of constant radius, a hyperbolic rotation involves points sliding around a hyperbola $(\cosh\alpha, \sinh\alpha)$. Additionally, while rotations preserve the dot product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + x_2 y_2$, the hyperbolic rotation preserves the Lorentz-Minkowski product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 - x_2 y_2$.

Note that performing two successive 'boosts' by rapidities α_1 and α_2 would be equivalent to performing only one transformation by rapidity $\alpha_1 + \alpha_2$. The Lorentz transform forms a group!

See here for an excellent resource on the derivation of the Lorentz transformation: `https://physics.umd.edu/~yakovenk/teaching/Lorentz.pdf`

7.3 Minkowski Diagrams

A Minkowski diagram, or spacetime diagram, is a way to graphically represent the Lorentz transform. It represents space and time as seen by a particular frame, considered to be at rest. The horizontal x-axis and vertical time axis are orthogonal in our rest frame. The time axis is typically shown as ct, which can be thought of as a regular time axis, but scaled to elicit some useful traits in our plot.

Straight lines on this plot represent inertial objects, and the further the slope deviation from the nearest orthogonal axis, the faster the object. Parallel lines on a spacetime diagram represent objects moving at 0 relative velocity. Vertical lines indicate objects at rest in the frame, since they are parallel to the ct axis. Points lying on a horizontal line for our frame represent simultaneous events, as seen by our observer.

Two critical lines on our plot are $x = \pm ct$, which denote the light cone, i.e. the world path of light in our frame. By the postulates of relativity, c is invariant in all frames, so the slope of this line is a constant 45° irregardless of frame. These lines segment the diagram into four regions:

- The two triangular quadrants directly above and below the origin represent the causal past (on the negative time axis) and the causal future (on the positive time axis);
- The remaining quadrants to the left and right of the origin represent 'elsewhere,' meaning areas which are inaccessible to the origin's present.

When mapping an objects position and time coordinates from $S \to S'$, we use a Lorentz transformation. Depending on the relative velocity between the two frames, we will see varying forms of time dilation and length contraction. On a spacetime diagram, a Lorentz boost tilts the coordinate axes by an angle $\alpha = \tanh^{-1}\frac{v}{c}$. Both the ct and x axes are squeezed symmetrically towards the light cone by this same angle, which preserves the light cone, i.e. the speed of light in all frames.

However, tilting both axes results in changes to the time and length measurements in our new frame. Consider a rod at rest in S, with proper length L_0. The ends of the rod are represented by two vertical lines at $x = 0$ and $x = L_0$ in our S frame. Now consider a frame S' moving with

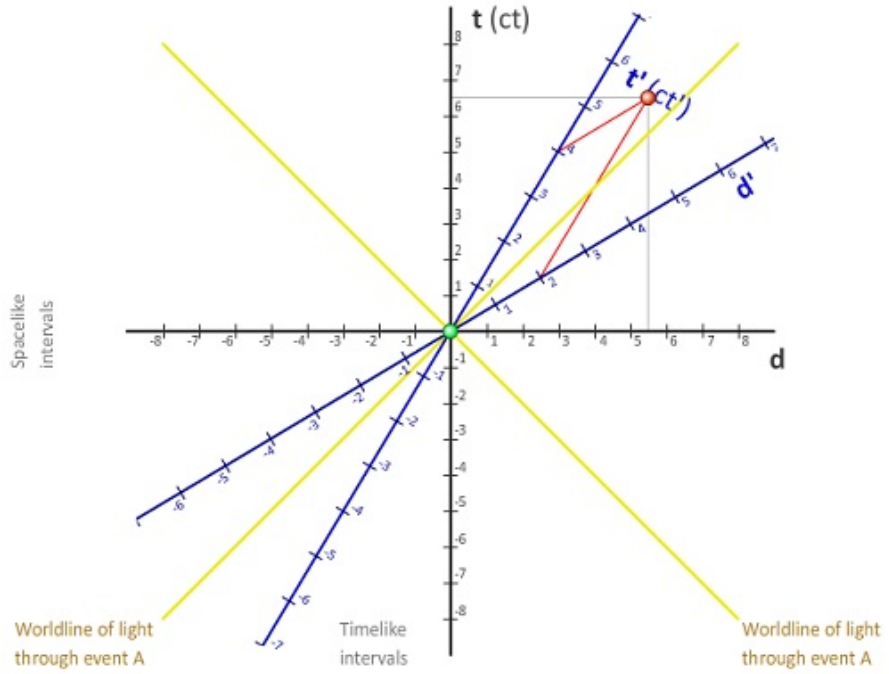


Figure 9: A simple visual illustrating the fundamental parts of a spacetime, or Minkowksi diagram.

speed v relative to S with rapidity α. We transform to S' by squeezing our ct' and x' axes by an angle of α with respect to the orthogonal axes in S. The measured length of the rod in S' will be $L = \frac{L_0}{\gamma} < L_0$. The axes squeeze towards the light cone, so the intersection points squeeze closer.

At first glance it seems like we've lost something here, since symmetry implies that for a rod of proper length L_0' at rest in S' will appear shorter in S moving at speed $-v$ relative to S'. Does this mean that our rod paradoxically gets shorter and shorter as we swap frames? No! Each frame sees the other frame's rod as contracted by γ, since the concept of length requires measuing the positions of the endpoints simultaneously. However, our Lorentz transformation tilts our simultaneity line, redefining what it means for simultaneity. Imagine a horizontal line on a Minkowski diagram for S. When tilting our axes, we measure the line to be shorter because we end up projecting only a component of the 'full' proper length onto the new axis. However, in reverse, we still end up projecting only a component of the 'full' proper length L_0' when remeasuring our rod in S, which explains our 'paradox'.

Note that the axes must each expand when doing this transformation such that each point slides along a hyperbola. The unit lengths of our two coordinate axes are warped by:

$$U' = U\sqrt{\frac{1+\beta^2}{1-\beta^2}}.$$

Drawing a line through our event parallel to the ct' axis, and finding its intersection with the x' axis, will give the observed x' coordinate (and vice versa for the ct' coordinate). This also explains our length contraction phenomenon!

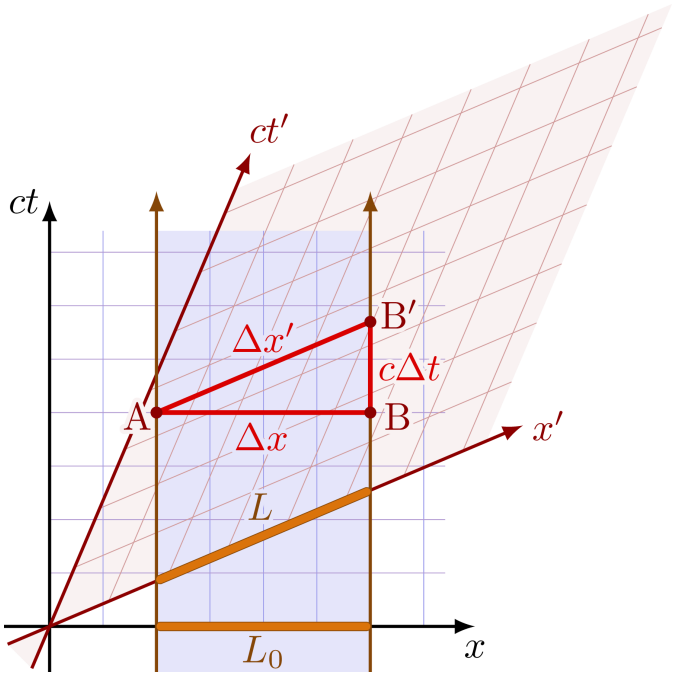


Figure 10: A diagram illustrating how a rod at rest in S will appear shorter in moving S'. Count the unit cells!

We know that the simultaneity of two events in our rest frame will not hold in our transformed frame. The constant time slices in our new frame lie on a line parallel with our new x' axis, at some angle α from our original x axis.

Here is an incredible reference that uses graphical techniques to discuss the subject! `https://tikz.net/relativity_minkowski_diagram/`

7.4 4 Vectors

We define the proper time experience by a particle to be:

$$\Delta\tau = \frac{\Delta s}{c}.$$

Which constitutes the minimum time interval that can be measured between two events, corresponding with an observer in the rest frame of the particle so: It is equivalently the time measured by a clock on the particle's world line, or the arc length on a Minkowski diagram. Expanding our definition, we can perform algebra showing:

$$d\tau^2 = dt^2 - \frac{d\boldsymbol{x}^2}{c^2} = dt\sqrt{1 - \frac{u^2}{c^2}}.$$

We note that:

$$\frac{dt}{d\tau} = \gamma.$$

This allows us to 'work in the particle's time' using parameterizations of τ with the 4-vector:

$$X(\tau) = \begin{bmatrix} ct(\tau) \\ \boldsymbol{x}(\tau) \end{bmatrix}.$$

And then the 4-velocity:

$$U = \frac{dX}{d\tau} = \begin{bmatrix} c\frac{dt}{d\tau} \\ \frac{d\boldsymbol{x}}{d\tau} \end{bmatrix} = \gamma \begin{bmatrix} c \\ \boldsymbol{u} \end{bmatrix}.$$

Where $\boldsymbol{u} = d\boldsymbol{x}/dt$. This results in a nice symmetry where if frame S measures 4-vector X and 4-velocity U, then frame S' will measure 4-vector ΛX and 4-velocity ΛU. This works since $\Delta\tau$ is invariant for all users, as it 'transforms' time back into the particles frame. Note that our 4-vectors are constrained to conserve the Minkowski metric.

Calculating the norm of the 4-velocity:

$$U \cdot U = \gamma\left(c^2 - u^2\right) = c^2.$$

We can define the 4-momentum as:

$$P = mU = \begin{bmatrix} mc\gamma \\ m\gamma\boldsymbol{u} \end{bmatrix}.$$

Where the relativistic generalization of our 3-momentum is:

$$\boldsymbol{p} = m\gamma\boldsymbol{u}.$$

As $u \to c$, we note that $p \to \infty$. Examining the time component of P using Taylor expansion:

$$P^0 = \frac{mc}{\sqrt{1-\beta^2}} = \frac{1}{c}\left(mc^2 + \frac{1}{2}mu^2 + \dots\right) = \frac{E}{c}.$$

Again, as $u \to c$, $E \to \infty$. For a stationary particle, all of the energy is contained in the rest mass $E = mc^2$.

Let's use the invariance of the inner product $P \cdot P$ in all frames to derive another relationship. In the particle's rest frame:

$$P \cdot P = m^2c^2.$$

While in a frame moving at speed u with respect to the rest frame:

$$P \cdot P = \frac{E^2}{c^2} - \boldsymbol{p}^2.$$

By the Minkowski metric. Therefore, we immediately get:

$$E^2 = m^2c^4 + \boldsymbol{p}^2c^2.$$

For massless particles, we can then see:

$$E^2 = p^2c^2 \implies E = pc.$$

Or equivalently that $P \cdot P = 0$, which since $P \cdot P \propto ds^2$ and $ds^2 = 0$ on the light cone, means that the particles lie on a light ray.

Here is a good resource outlining many of these results: `https://www.damtp.cam.ac.uk/user/tong/relativity/seven.pdf`