Continuous Time Signals and Systems

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Contents

1 Continuous Time Signals

1.1 Classifying Signals

Lecture 1

2026-01-05

Definition 1. A cts-time signal is a function that carries information, with time as the independent variable.

$$f(t), \quad X(t).$$

We will deal with scalar, or *single-channel* signals, not vector, or *multi-channel* signals.

The independent variable can be a single variable, a *one-dimensional* signal, or can be a vector, a *multi-dimensional* signal. Note that cts-time signals have an independent variable $t \in \mathbb{R}$ over some specified interval, while discrete-time signals have independent variable that can only take on certain values, e.g. $t \in \mathbb{Z}$.

2026-01-07

Lecture 2

Definition 2. A causal signal $f(t) = 0 \forall t < 0$.

Definition 3. An anti-causal signal $f(t) = 0 \forall t > 0$

A non-causal signal is anything that is anything not causal or anti-causal.

Definition 4. An odd signal is antisymmetric, and satisfies $f_o(t) = -f_o(-t) \, \forall t$. An even signal is symmetric about the y-axis, $f_e(t) = f_e(-t) \, \forall t$.

Note that even and odd signals satisfy some operational properties:

- The sum of two like-signals is the same type
$$f_e \pm f_e = f_e, \quad f_o \pm f_o = f_o.$$
- The product of two like-signals is even
$$f_o \times f_o = f_e, \quad f_e \times f_e = f_e.$$
- The product of an odd number of odd functions with even functions is odd
$$f_e \times f_o \times f_e = f_o.$$

These properties can be verified by defining our final function $g(t)$, and then comparing $g(t)$ with $g(-t)$.

Theorem 1. Every signal can be written as the sum of an even signal and an odd signal.

Proof:

$$\begin{aligned} f(t) &= \frac{1}{2}f(t) + \frac{1}{2}f(t) + \frac{1}{2}f(-t) - \frac{1}{2}f(-t) \\ &= \frac{1}{2}(f(t) + f(-t)) + \frac{1}{2}(f(t) - f(-t)). \end{aligned}$$

Which is the sum of an even function $f_e(t) = \frac{1}{2}(f(t) + f(-t))$and an odd function $f_o(t) = \frac{1}{2}(f(t) - f(-t))$.

Theorem 2. Integrals of even and odd functions satisfy:

$$(i) \int_{-a}^{a} f_e(t)\, dt = 2\int_{0}^{a} f_e(t)\, dt \qquad (ii) \int_{-a}^{a} f_o(t)\, dt = 0.$$

Proof of (i): We can split out integral and make a substitution:

$$\int_{-a}^{a} f(t)\, dt = \int_{-a}^{0} f(t)\, dt + \int_{0}^{a} f(t)\, dt.$$

Let $x = -t, dx = -dt$, the first integral on the right-hand side becomes:

$$-\int_{a}^{0} f(-x)\, dx = \int_{0}^{a} f(x)\, dx.$$

By the properties of even functions.

The proof of (ii) uses the same method, however the properties of odd functions mean an additional negative is added to one of the terms, cancelling them out.

Definition 5. $f(t)$ is periodic if $\exists T, T \in \mathbb{R}$ such that:

$$f(t) = f(t+T).$$

The smallest number T that satisfies the definition is the period.

Note that since $f(t) = f(t+T) \implies f(t) = f(t+kT), k \in \mathbb{Z}$ by an induction-type logic. However, a signal must be defined $\forall \mathbb{R}$ to be periodic! Recall that we can find a 2L-periodic *extension* of a non-periodic function with Fourier series.

Lecture 3

2026-01-09

Definition 6. We define the energy of a signal as:

$$E_t = \int_{-\infty}^{\infty} |f(t)|^2\, dt < \infty.$$

Where $|f(t)|^2 = f(t)\overline{f}(t)$ for complex signals, which collapses to $f(t)^2$ for real signals.

This is the measure of the 'size' of a signal, but cannot be used in the case of signals with non-finite energy. For example, $f(t) = \cos\omega t$ has infinite energy, as $\int_{-\infty}^{\infty} \cos^2 \omega t\, dt$ diverges.

Note. The energy of the signal is related to energy in physics, in the way that we associate energy with the square of the wave amplitude. △

Definition 7. We define the average power of a signal as:

$$P = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} |f(t)|^2\, dt.$$

Note. Signals with finite energy are called energy signals. Signals with *non-zero* finite power are called power signals. These definitions are mutually exclusive, since finite energy $\implies$ 0 power. △

For a periodic signal, $f(t) = f(t + nT), n \in \mathbb{Z}$, $\lim_{\tau \to \infty} z = \lim_{n \to \infty} nT$.

$$P_t = \lim_{\tau \to \infty} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} |f(t)|^2 \, dt = \lim_{n \to \infty} \frac{1}{nT} \int_{-\frac{nT}{2}}^{\frac{nT}{2}} |f(t)|^2 \, dt = \lim_{n \to \infty} \frac{1}{nT} \left(n \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 \, dt \right).$$

Since the power of n periods is the same as n times the power of 1 period.

Definition 8. For a periodic signal, the power collapses to:

$$P_t = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 \, dt.$$

Definition 9. The root mean square (RMS) value of a signal is defined as:

$$\mathrm{RMS}_t = \sqrt{P_t}.$$

Note. In general, by trigonometric identities,

$$f(t) = \sum_i A_i \cos(\omega_i t + \theta_i) \implies P_t = \sum_i \frac{A_i^2}{2}.$$

△

2026-01-12 **Lecture 4**

1.2 Transformations of the Independent Variable

Definition 10. A **time reversal** is a reflection across the y-axis

$$y(t) = f(-t).$$

Definition 11. Time scaling maps the input signal to a horizontally scaled (along independent variable axis) signal

$$y(t) = f(at).$$

Which is stretched by a factor of $\frac{1}{a}$

- $|a| < 1$: corresponds with a stretch,
- $|a| > 1$: corresponds with a compression,
- $a < 0$: corresponds with a scale and reversal;

Definition 12. Time shifting slides the signal along the horizontal axis

$$y(t) = f(t - b).$$

- $b > 0$: corresponds with a right shift,
- $b < 0$: corresponds with a left shift

Note that in the case of combined operations, the order of operations matters! To shift by b and scale by a, either:

- First shift, then scale, applying the scale only to t;
- First scale, then shift by $\frac{b}{a}$ by replacing at with $a\left(t - \frac{b}{a}\right)$ since the a will apply to the b!

For example, to draw $y(t) = f(2t+3)$, we would first shift left 3, then compress by $\frac{1}{2}$. Alternatively, we could compress by $\frac{1}{2}$, then shift left $\frac{3}{2}$.

1.3 Elementary Signals

The Unit Step

The Heaviside function is a shifted and scaled arctan curve, which approaches 0 as $x \to -\infty$, and approaches 1 as $x \to \infty$.

$$H(t) = \frac{1}{2} + \frac{1}{\pi}\arctan(\lambda t), \lambda \in \mathbb{R}^+.$$

We can similarly use $H(t) = \frac{1}{2} + \frac{1}{2}\tanh(\lambda t), \lambda \in \mathbb{R}^+$.

Definition 13. If we take $\lim_{\lambda\to\infty} H(t)$, we recover the **unit step function**, as the arctan curve gets narrower and narrower.

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}.$$

This function is continuous everywhere except $t = 0$ where it is not defined (subject to argument).

This is really handy to represent causal or anti-causal signals by multiplying by a (reversed) unit step function. We can also use this to create the rectangular pulse, using the addition of two relatively shifted unit step functions.

Definition 14. The unit rectangular pulse is defined as

$$p_d(t) = \begin{cases} 1 & |t| < \frac{1}{2}d \\ 0 & |t| > \frac{1}{2}d \end{cases} = u\left(t + \frac{1}{2}d\right) - u\left(t - \frac{1}{2}d\right).$$

Where d is the width of the even signal.

Lecture 5

2026-01-14

Let us differentiate our definition of the unit step function:

$$\frac{d}{dt}H(t) = \frac{1}{\pi}\left(\frac{\lambda}{1+\lambda^2 t^2}\right).$$

Observe that as $\lambda \to \infty$, the derivative goes to 0 everywhere except at $t = 0$, where it goes to infinity. It is convenient to introduce $\sigma = \frac{1}{\lambda} \implies \sigma \in \mathbb{R}^+$.

Definition 15. The unit impulse, or Dirac Delta function, is defined as:

$$\delta\left(t\right) = \lim_{\sigma \to 0^{+}} \frac{1}{\pi}\left(\frac{\sigma}{\sigma^{2} + t^{2}}\right).$$

Which is 0 everywhere, except at $t = 0$, where it is discontinuous, and goes to ∞.

$$\delta\left(t\right) = \begin{cases} 0 & t \neq 0 \\ \int_{-\infty}^{\infty} \delta\left(t\right) dt = 1 & \end{cases}.$$

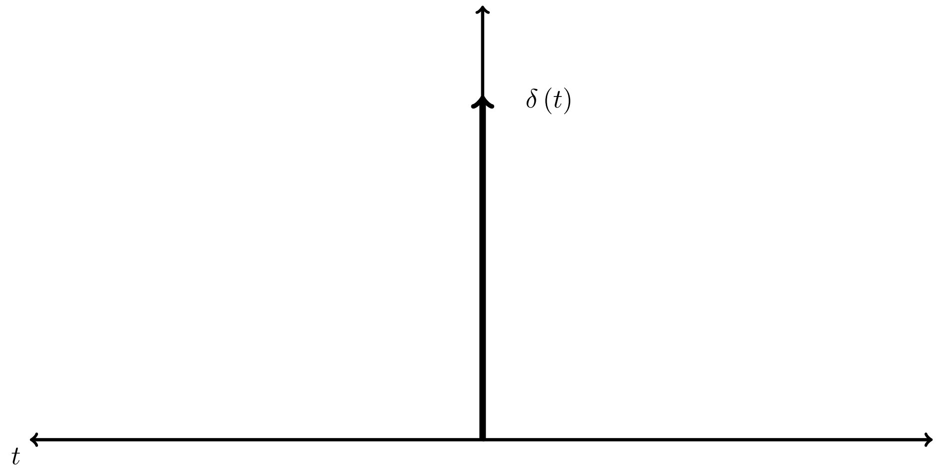

Figure 1: Visualization of the unit impulse

Note. Observe that the integral from $-\infty \to \infty$ of $\delta\left(t\right) = 1$:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = \lim_{\sigma \to 0^{+}} \frac{1}{\pi} \arctan\left(\frac{t}{\sigma}\right) |_{-\infty}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1 \quad \forall \sigma.$$

△

The Dirac Delta signal is actually not a function, rather a distribution.

Properties of $\delta\left(t\right)$

- Multiply by a constant: scales the impulse as $A \int_{-\infty}^{\infty} \delta\left(t\right) dt = A$;
- Shift the impulse $\delta\left(t - T\right)$ shifts T units to the right;
- Multiply the impulse by a function $\phi\left(t\right)$:

$$\phi\left(t\right)\delta\left(t - T\right) = \begin{cases} 0 & t \neq T \\ \phi\left(T\right)\delta\left(0\right) & t = T \end{cases}.$$

Provided that ϕ is continuous at T

- The ‘sifting function’ property, for some ϕ continuous at $t = T$, is an application of these properties:

$$\int_{-\infty}^{\infty} \phi(t)\,\delta(t-T)\,dt = \phi(T)\,.$$

Theorem 3.

$$\delta(t) = \frac{d}{dt}u(t)$$
$$u(t) = \int_{-\infty}^{t} \delta(t')\,dt'.$$

This arises by the two functions sharing a limit definition of $H(t)$, differing by a derivative. We can swap the order of limit and differentiation to get these relations.

$$\delta(t) = \frac{d}{dt}\lim_{\sigma\to 0} H(t) = \frac{d}{dt}u(t)\,.$$

Lecture 6 — 2026-01-16

Definition 16. The sinc function is defined as:

$$\operatorname{sinc}(t) = \frac{\sin t}{t} \quad \forall t.$$

Sinc has zeros at $\pm\pi, \pm 2\pi, \ldots \pm k\pi, k \in \mathbb{Z}$.

2 Systems

Definition 17. A system is an operator that transforms input signals into output signals.

$$y = H(f)\,, \quad f(t) \overset{H}{\to} y(t)\,.$$

Definition 18. A system has memory if the output $y(t_0)$ depends on the input value $f(t)$ at some time $t \neq t_0$. Otherwise, the system is memoryless.

For example, the voltage across a capacitor is not memoryless, as it depends on the value of the current for times $t < t_0$.

Definition 19. A system is causal if the output $y(t_0)$ depends on values of the input $f(t)$ for $t \leq t_0$, and does not depend on future inputs. Otherwise, a system is anti-causal.

Lecture 7 — 2026-01-19

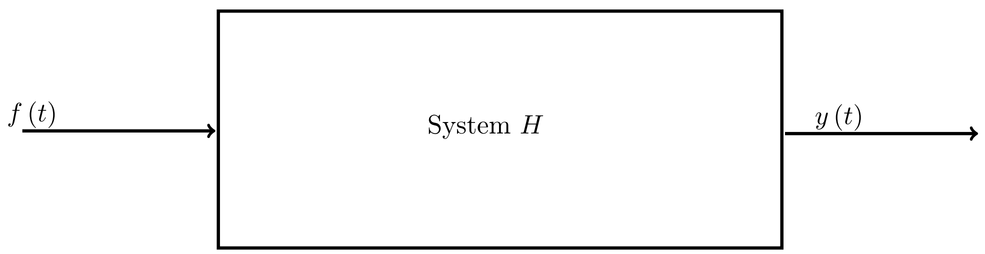

Figure 2: Black box visualization of a system

Definition 20. A system H is linear if it preserves the operations of signal addition and scaling, or superposition and homogeneity:

$$H(f_1(t)) = y_1(t)$$
$$H(f_2(t)) = y_2(t)$$
$$H(af_1 + bf_2) = ay_1 + by_2, \forall a, b \in \mathbb{R}.$$

Example. Take the system $y(t) = [f(t)]^k, k \in \mathbb{R}$.

Observe that:

$$a_1 y_1 + a_2 y_2 = a_1 f_1^k + a_2 f_2^k \neq H(a_1 f_1 + a_2 f_2) = (a_1 f_1 + a_2 f_2)^k.$$

Therefore our system is not linear. $\Diamond$

Example. Take the system $H(f(t)) = \frac{df}{dt}$. Observe that differentiation is a linear operator, so the system will be linear: For $y_1 = H(f_1)$ and $y_2 = H(f_2)$, observe that:

$$H(a_1 f_1 + a_2 f_2) = a_1 \frac{df_1}{dt} + a_2 \frac{df_2}{dt} = a_1 y_1 + a_2 y_2.$$

$\Diamond$

Example. Take a system $H : f(t) \to y(t)$, where our system is governed by the differential equation:

$$\frac{dy}{dt} + by = kf.$$

We can test for linearity by simply taking a linear combination of f_1 and f_2, where $H(f_1) = y_1$

and $H(f_2) = y_2$. Note that y_1 and y_2 satisfy the differential equation:

$$\begin{aligned}\frac{dy_1}{dt} + by_1 &= kf_1\\ \frac{dy_2}{dt} + by_2 &= kf_2.\end{aligned}$$

Taking our linear combination:

$$\frac{d}{dt}(a_1y_1 + a_2y_2) + b(a_1y_1 + a_2y_2) = k(a_1f_2 + a_2f_2).$$

If this is true, then H is linear.

$$a_1\frac{dy_1}{dt} + a_2\frac{dy_2}{dt} + ka_1y_1 + ka_2y_2 = ka_1f_1 + ka_2f_2.$$

This is the sum of our two assumptions, each multiplied by a_1 and a_2 respectively. ♢

Example. System $H : f(t) \to y(t)$ governed by:

$$y\frac{dy}{dt} + by = f.$$

Take $H(f_1)$ satisfied by y_1 and $H(f_2)$ satisfied by y_2. Multiply our y_1 equation by a_1 and our y_2 equation by a_2, and sum:

$$\begin{aligned}a_1^2y_1\frac{dy_1}{dt} + a_1by_1 &= a_1f_1\\ a_2^2y_2\frac{dy_2}{dt} + a_2by_2 &= a_2f_2\\ a_1^2y_1\frac{dy_1}{dt} + a_2^2y_2\frac{dy_2}{dt} + b(a_1y_1 + a_2y_2) &= a_1f_1 + a_2f_2.\end{aligned}$$

Note that this does not match our expected value, both superposition and homogeneity fails since it is a non-linear operator. ♢

Definition 21. A system $H : f(t) \to y(t)$ is time invariant if a time shift in the input variable results in the same time shift in the output variable.

$$H : f(t) \to y(t) \implies H : f(t-T) \to y(t-T).$$

The time-dependence of the system must be fully self-contained in the input, but this is necessary, not sufficient.

Example. Take system $H : f(t) \to y(t)$ given by $y(t) = (f(t))^2$.

Observe that $y(t-T) = f(t-T)^2$. Applying a new function to our system:

$$H(f(t-T)) = f(t-T)^2.$$

Since our two results match, our system is time invariant. ♢

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Test for time invariance:

- First find $y_1(t)$ in terms of $f_1(t)$;
- Shift our output $y_1(t-T)$;
- Take $f_2 = f_1(t-T)$ and find y_2;
- If $y_1(t-T) = y_2(t)$, the system is time invariant.

Definition 22. A system $H : f(t) \to y(t)$is bounded input-bounded output (BIBO) stable if every bounded input results in a bounded output:

$$|y(t)| \leq B < \infty \quad \forall \quad |f(t)| \leq A < \infty.$$

3 Linear Time Invariant Systems

Recall that an LTI system is a linear operator that maps shifts in inputs to shifts in outputs.

Definition 23. For $f_1(t)$ and $f_2(t)$, the convolution is defined as:

$$(f_1 * f_2)(t) = \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau)\, d\tau.$$

2026-01-23 **Lecture 9**

Example. Find $y(t) = (u(t) - u(t-4)) * (e^{-t}u(t))$.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} (u(\tau) - u(\tau-4))\, e^{-t+\tau} u(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} u(\tau)\, u(t-\tau)\, e^{\tau-t}\, d\tau - \int_{-\infty}^{\infty} u(\tau-4)\, u(t-\tau)\, e^{\tau-t}\, d\tau \\ &= y_1 - y_2. \end{aligned}$$

Simplifying y_1 with properties of $u(t)$:

$$\begin{aligned} y_1 &= \int_{0}^{\infty} u(t-\tau)\, e^{\tau-t}\, d\tau \\ &= e^{-t} \int_0^t e^{\tau}\, d\tau, \quad \tau < t, t > 0 \\ &= \begin{cases} 1 - e^{-t} & t > 0 \\ 0 & t < 0 \end{cases}. \end{aligned}$$

Now for y_2:

$$\begin{aligned} y_2 &= e^{-t} \int_4^{\infty} u(t-\tau) e^{\tau} \, d\tau \\ &= e^{-t} \int_4^{t} e^{\tau} \, d\tau, t > 4 \\ &= \begin{cases} 1 - e^{4-t} & t > 4 \\ 0 & t < 4 \end{cases}. \end{aligned}$$

Therefore:

$$y(t) = \left(1 - e^{-t}\right) u(t) - \left(1 - e^{4-t}\right) u(t-4).$$

Using the unit step function to replace our domain restrictions. ♢

Properties of convolution

- The convolution is commutative: $f_1 * f_2 = f_2 * f_1$.

  Proof

  $$f_1 * f_2 = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) \, d\tau.$$

  Define $t - \tau = x \implies d\tau = -dx$ This substitution yields:

  $$f_1 * f_2 = -\int_{\infty}^{-\infty} f_1(t-x) f_2(x) \, dx = f_2 * f_1.$$

- The convolution is distributive: $f_1 * (f_2 + f_3) = f_1 * f_2 + f_1 * f_3$.

  Proof

  Just distribute the function through the integrand and split into two integrals.

- The convolution is associative: $(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$.
- Convolutions propagate time shifts. If $y = f_1 * f_2$, then $f_1 * f_2(t-T) = y(t-T)$, $f_1(t-T) * f_2 = y(t-T)$, and $f_1(t-T_1) * f_2(t-T_2) = y(t-T_1-T_2)$.

  Proof

  First examine:

  $$f_1 * f_2(t-T) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-T-\tau) \, d\tau = y(t-T).$$

  The second idea follows from commutativity of convolution. And the third idea the same way.

- The convolution with the impulse $f * \delta = f(t)$

  Proof

  $$f * \delta = \int_{-\infty}^{\infty} f(\tau) \delta(t-\tau) \, d\tau = f(t).$$

- The convolution of causal signals $f_1 * f_2$, $f_1 = f_2 = 0 \forall t < 0$ is $\int_0^t f_1(t) f_2(t-\tau)\, d\tau, t > 0$.

  Proof

  Take the convolution and realize that the integrand is $0 \forall \tau < 0$, and $\forall \tau > t$.

Example. Find $y(t) = (u(t) - u(t-4)) * (e^{-t} u(t))$.

By distributivity:

$$y = u(t) * e^{-t} u(t) - u(t-4) * e^{-t} u(t).$$

For a previous example, we find the first term to be $(1 - e^{-t}) u(t)$. Using our time shifting property, the second term is trivially $\left(t - e^{-(t-4)}\right) u(t-4)$, which matches the result in our previous example. ♢

2026-01-26

Lecture 10

3.1 Properties of LTI Systems

A convolution is a linear time invariant system. When the convolution $H : f(t) \to y(t)$ acts on $\delta(t)$, the result $h(t) = H(\delta(t))$ is called the *impulse* of the system.

Theorem 4. Given an LTI system H with impulse response $h(t)$, the system output to an arbitrary input $f(t)$ is given by:

$$y(t) = f(t) * h(t).$$

Proof

Recall that any function $f(t)$ convoluted with the delta function is itself:

$$f(t) = f * \delta = \int_{-\infty}^{\infty} f(\tau) \delta(t-\tau)\, d\tau.$$

Define $y(t) = H[f(t)]$. If the system is linear, then we can distribute it through the integral, an infinite summation:

$$y(t) = H\left[\int_{-\infty}^{\infty} f(\tau) \delta(t-\tau)\, d\tau\right] = \int_{-\infty}^{\infty} H[f(\tau) \delta(t-\tau)]\, d\tau.$$

We can pull $f(\tau)$ outside of the operator since it is not a function of t. If the system is time invariant, then any shifts in the input are mapped directly to the output.

$$y(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau)\, d\tau = f * h.$$

Theorem 5. An LTI system H is memoryless $\iff h(t) = k\delta(t)$.

Proof

Take the output of the system $H : f \to y$

$$y(t) = h * f \implies y(t_0) = h * f|_{t=t_0}.$$

Evaluate the definition of the convolution at $t = t_0$:

$$y(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau \implies y(t_0) = \int_{-\infty}^{\infty} f(\tau)\, h(t_0-\tau)\, d\tau.$$

If H is memoryless, then $y(t_0) = kf(t_0)$. This is true $\iff$ $h(t_0 - \tau)$ behaves like the delta function, $h(t) = k\delta(t)$

Theorem 6. An LTI system H is causal $\iff$ $h(t) = 0 \forall t < 0$, i.e. $h(t)$ must itself be causal.

Proof

Examine the output $y = H[f]$.

$$y(t) = f * h = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau.$$

Causality of a system means $y(t_0)$ cannot depend on future values of f.

$$y(t_0) = \int_{-\infty}^{t_0} f(\tau)\, h(t_0-\tau)\, d\tau + \int_{t_0}^{\infty} f(\tau)\, h(t_0-\tau)\, d\tau.$$

The second integral must be 0 for the system to be causal, meaning $h(t_0 - \tau) = 0 \forall \tau \in (t_0, \infty)$. This implies that $h(t) = 0$ for all negative inputs.

Theorem 7. An LTI system H is BIBO stable $\iff$ $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$.

Proof

Let's assume that $h(t) : \int_{-\infty}^{\infty} |h(t)|\, dt < M < \infty$ for some $M \in \mathbb{R}$. For $y = f * h$, we can find the magnitude:

$$|y| = |\int_{-\infty}^{\infty} h(\tau)\, f(t-\tau)\, d\tau| \leq \int_{-\infty}^{\infty} |h(\tau)||f(t-\tau)|\, d\tau.$$

Since the input is always bounded, $|f(t)| \leq A$, or $|f(t-\tau)| \leq A$, then the output is bounded as:

$$|y(t)| \leq MA.$$

If the original assumption is satisfied. This condition is sufficient.

For a necessary condition, let's assume that h is not bounded, i.e. $\int_{-\infty}^{\infty} |h(t)|\, dt \to \infty$. For any value $N \in \mathbb{R}^+, \quad \exists t_0 :$

$$\int_{-\infty}^{t_0} |h(t)|\, dt > N.$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, f(t-\tau)\, d\tau \implies y(t_0) = \int_{-\infty}^{t_0} h(\tau)\, f(t_0-\tau)\, d\tau.$$

Define f as a bounded function:

$$f(t_0 - \tau) = \begin{cases} 1 & h(\tau) > 0 \\ -1 & h(\tau) < 0 \end{cases}.$$

This forces $h(\tau) > 0 \forall \tau$, which collapses to our initial property of the integral over h. We note that this results in an unbounded output regardless, meaning a bounded integral over h results in an unbounded output.

$$y(t_0) = \int_{-\infty}^{t_0} |h(\tau)| \, d\tau.$$

2026-01-28 **Lecture 11**

4 Fourier Series

4.1 Inner Product and Orthogonality

Definition 24. The inner product between two complex vectors $\boldsymbol{u}, \boldsymbol{v}$ as any operation satisfying the following properties:

1. Conjugate symmetric:
$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}.$$
2. Homogeneity:
$$\langle c\boldsymbol{u}, \boldsymbol{v} \rangle = c\langle \boldsymbol{u}, \boldsymbol{v} \rangle, c \in \mathbb{C}.$$
3. Positivity:
$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geq 0, \langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 \iff \boldsymbol{u} = 0.$$
4. Additivity:
$$\langle \boldsymbol{u} + \boldsymbol{w}, \boldsymbol{v} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{w}, \boldsymbol{v} \rangle.$$

For two vectors $x_1, v_2 \in \mathbb{R}$, a valid inner product is the dot product $\langle x_1, x_2 \rangle = x_1 \cdot x_2 = |x_1||x_2| \cos\theta$. Take two orthonormal basis vectors $\hat{e}_1, \hat{e}_2$ spanning $\mathbb{R}^2$. Taking the dot product $\hat{e}_i \cdot \boldsymbol{u}$ will 'extract' the component of $\boldsymbol{u}$ parallel to $\hat{e}_i$ since each $\hat{e}_i$ are of unity magnitude. Additionally, any $\boldsymbol{u} \in \mathbb{R}^2$ can be represented as a linear combination of $\sum_i a_i \hat{e}_i$.

The function space is an infinite dimensional vector space. In the function space, we define the inner product between f_1 and f_2 to be:

$$\langle f_1, f_2 \rangle = \int_a^b f_1(t) \overline{f_2}(t) \, dt.$$

We include the complex conjugate to cover potential complex-valued functions.

This implies that:

- Two functions f_1 and f_2 are orthogonal over an interval if their inner product over that interval is 0.

2026-01-30 **Lecture 12**

Take a generalized $\sin n\omega_0 t$ and $\cos n\omega_0 t, n \in \mathbb{Z}^+$. The fundamental period of these sinusoids is $T = \frac{2\pi}{\omega_0}$ derived from the fundamental angular frequency ω_0. We can find higher order harmonics of these functions by increasing n, resulting in the set:

$$S = \{1, \cos(\omega_0 t), \cos(2\omega_0 t), \ldots, \sin(\omega_0 t), \sin(2\omega_0 t), \ldots\}.$$

Definition 25. The set S is an orthogonal set on the interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$.

This implies that:

1. Cosines of different frequencies are orthogonal:
$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(n\omega_0 t\right)\cos\left(m\omega_0 t\right) dt = \begin{cases} 0 & n \neq m \\ \frac{T}{2} & n = m \neq 0 \end{cases}.$$
2. Sines of different frequencies are orthogonal:
$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin\left(n\omega_0 t\right)\sin\left(m\omega_0 t\right) dt = \begin{cases} 0 & n \neq m \\ \frac{T}{2} & n = m \neq 0 \end{cases}.$$
3. Cosines and sines are always orthogonal:
$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(n\omega_0 t\right)\sin\left(m\omega_0 t\right) dt = 0 \forall n, m.$$
4. Cosines and sines integrated over a period is 0:
$$\int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \cdot \cos\left(n\omega_0 t\right) dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \cdot \sin\left(n\omega_0 t\right) dt = 0, \forall n.$$

Let's prove some of these properties. The first property can be proven using the trigonometric identity $\cos A \cos B = \frac{1}{2}\left(\cos\left(A+B\right) + \cos\left(A-B\right)\right)$

4.2 Fourier Series

Definition 26. The trigonometric Fourier series is an infinite sum of harmonics of sinusoids as follows:
$$f\left(t\right) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(n\omega_0 t\right) + b_n \sin\left(n\omega_0 t\right)\right].$$
Where $\omega_0 = \frac{2\pi}{T}$ is the fundamental angular frequency of f.

Lecture 13

2026-02-02

Definition 27. The Trigonometric Fourier Series (TFS) converging to some periodic function $f\left(t\right) = f\left(t+T\right), \omega = \frac{2\pi}{T}$, is defined as:
$$f\left(t\right) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(n\omega_0 t\right) + b_n \sin\left(n\omega_0 t\right)\right].$$
With:
$$a_0 = \frac{1}{T}\int_T f\left(t\right) dt, \quad a_n = \frac{2}{T}\int_T f\left(t\right)\cos\left(n\omega_0 t\right) dt, \quad b_n = \frac{2}{T}\int_T f\left(t\right)\sin\left(n\omega_0 t\right) dt.$$

Derivation of equations

We can integrate both sides over a period T to extract our coefficient a_0 :

$$\int_T f(t)\,dt = \int_T a_0\,dt + \int_T \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right] dt.$$

Since our sinusoid terms go to 0, we get:

$$a_0 = \frac{1}{T}\int_T f(t)\,dt.$$

We can multiply both sides by $\cos(m\omega_0 t)$ and integrate in order to extract our coefficient series a_n.

$$\int_T f(t)\cos(m\omega_0 t)dt = \int_T a_0 \cos(m\omega_0 t)dt + \int_T [a_n \cos(n\omega_0 t)\cos(m\omega_0 t) + b_n \sin(n\omega_0 t)\cos(m\omega_0 t)]dt.$$

Observe that our a_0 integral goes to 0 by integrating over an integer multiple of the period of a single sinusoid, and the b_n term goes to 0 by our orthogonality relations. For our a_n integral, when $m \neq n$, the integral goes to 0 by orthogonality relations. However, for $m = n$, the integral goes to $\frac{T}{2}$. Therefore, we can write our a_n as:

$$a_n = \frac{2}{T}\int_T f(t)\cos(n\omega_0 t)\,dt.$$

The b_n terms can be found similarly as:

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\omega_0 t)\,dt.$$

Note. It is important to note simplifications in the case of even or odd functions $f(t)$. An even f means the b_n formula integrates an odd function over symmetric bounds, so $b_n = 0 \forall n$. This results in a cosine-only series.

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos(n\omega_0 t)\,dt, \quad a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)\,dt.$$

Alternatively, an odd f means the a_n formula integrates an odd function over symmetric bounds, so $a_n = 0 \forall n$, producing a sine-only series.

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin(n\omega_0 t)\,dt.$$

△

2026-02-04 **Lecture 14**

The partial sum of a TFS is represented as:

$$S_k(t) = a_0 + \sum_{n=1}^{k}\left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right].$$

Definition 28. Representing a function $f(t)$ with a partial Fourier Series $S_k(t)$ will have a mean square error:

$$E_k = \frac{1}{T}\int_T (f(t) - S_k(t))^2\, dt.$$

Expanding this, noticing that cross terms go to 0 due to orthogonality relations, we can obtain the following theorem.

Theorem 8.

$$E_k = \frac{1}{T}\int_T f(t)^2\, dt - a_0^2 - \frac{1}{2}\sum_{n=1}^{k}\left(a_n^2 + b_n^2\right).$$

This shows that our E_k is the difference in power of the actual function to the partial series.

Since $E_k \geq 0$, we can get:

$$\frac{1}{T}\int_T f(t)^2\, dt \geq a_0^2 + \frac{1}{2}\sum_{n=1}^{k}\left(a_n^2 + b_n^2\right).$$

Theorem 9. If we consider $\{a_n\}$ and $\{b_n\}$ as series of coefficients, then:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0.$$

Proof

Manipulating our earlier inequality, we can observe that the mean-square of $f(t)$ over its period is bounded.

$$\frac{1}{T}\int_T f(t)^2\, dt \geq a_0^2 + \frac{1}{2}\sum_{n=1}^{k}\left(a_n^2 + b_n^2\right).$$

Therefore, as we take $k \to \infty$, we have an infinite series of positive terms which implies that it is bounded and monotonically increasing. For both of these to be true, both $a_n, b_n \to 0$ as $n \to \infty$.

Theorem 10. Parseval's Theorem: The power of $f(t) = f(t+T)$ is given by:

$$P_f = \frac{1}{T}\int_T f(t)^2\, dt = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

For the coefficients a_n and b_n in the TFS of f.

This quantifies the contribution of each harmonic to the overall power of the signal.

Proof

Using our inequality from earlier for E_k, we can show that $E_{k+1} = E_k + \frac{1}{2}\left(a_{k+1}^2 + b_{k+1}^2\right)$. Since $a_k, b_k \to 0$ as $k \to \infty$, then $E_{k+1} - E_k \to 0$ as $k \to \infty$, meaning as $k \to \infty$, Parseval's Theorem emerges from our inequality.

Note. We have to be careful about convergence of the TFS. We have shown that as $k \to \infty$ results the difference between S_k and f going to 0 in the norm (in the mean), i.e. the mean square error $E_k \to 0$. However, this does not imply point-wise convergence. There are two conditions for convergence in the mean:

$$f(t) = f(t+t), \quad \int_T |f(t)|\, dt < \infty.$$

Neither of which is continuity. As such, when f has a jump discontinuity, the TFS converges to the mean value of the left and right limits. △

Definition 29. Dirichlet Conditions for existence of the TFS for periodic $f(t) = f(t+T)$:

1. Weak Dirichlet Condition: a_0, a_n, b_n must be finite, which has the sufficient condition that $\int_T |f(t)|\, dt < \infty$. This gives convergence in the mean.
2. Strong Dirichlet Condition: $f(t)$ has a finite number of discontinuities, and has a finite number of maxima and minima.

   This gives point-wise convergence at every point where f is continuous and differentiable, and mean convergence at discontinuities.

2026-02-06

Lecture 15

Observe that we can obtain our trigonometric summation identity as:

$$\begin{aligned} e^{i(A+B)} &= e^{iA} e^{iB} \\ &= (\cos A + i \sin A)(\cos B + i \sin B) \\ &= \cos A \cos B - \sin A \sin B + i (\cos A \sin B + \sin A \cos B). \end{aligned}$$

This allows us to write a set of harmonics in the trigonometric Fourier Series as:

$$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) = c_n \cos(n\omega_0 t + \phi_n) = c_n (\cos(n\omega_0 t)\cos\phi_n - \sin(n\omega_0 t)\sin\phi_n).$$

This implies that:

$$\begin{aligned} a_n &= c_n \cos\phi_n \\ b_n &= -c_n \sin\phi_n \\ c_n &= +\sqrt{a_n^2 + b_n^2}\ (c_0 = a_0) \\ \phi_n &= \arctan -\frac{b_n}{a_n}. \end{aligned}$$

Definition 30. This results in the compact form of the TFS:

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n).$$

Where $c_n = \sqrt{a_n^2 + b_n^2}$ is the magnitude our coefficients, and $\phi_n = \arctan -\frac{b_n}{a_n}$ is the angle between them.

We can image this as a right angle triangle with a_n along the positive x-axis, and b_n along the positive y-axis. This stems from a_n and b_n being coefficients of orthogonal functions.

- Take $b_n = 0$: $a_n > 0 \implies \phi_n = 0, a_n < 0 \implies \phi_n = \pm\pi$
- Take $a_n = 0$: $b_n > 0 \implies \phi_n = -\frac{\pi}{2}$, $b_n < 0 \implies \phi_n = \frac{\pi}{2}$

Other cases can be obtained by analyzing the sign and geometric relationships of the coefficients.

We can plot the spectra of our compact form, which displays information about each harmonic of our series. Plotting c_n against ω is the frequency spectrum. Plotting ϕ_n against ω is the phase spectrum.

Lecture 16

2026-02-09

Using Euler's identity:

$$e^{j\theta} = \cos\theta + j\sin\theta, \quad e^{-j\theta} = \cos\theta - j\sin\theta.$$

We can represent the cosines and sines in our TFS as:

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

We can then write our TFS using these substitutions:

$$f(t) = a_0 \sum_{n=1}^{\infty} \left[\frac{a_n}{2} e^{jn\omega_0 t} + \frac{a_n}{2} e^{-jn\omega_0 t} + \frac{b_n}{2j} e^{jn\omega_0 t} - \frac{b_n}{2j} e^{-jn\omega_0 t}\right].$$

We can simplify this as:

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left[\frac{1}{2}(a_n - jb_n) e^{jn\omega_0 t} + \frac{1}{2}(a_n + jb_n) e^{-jn\omega_0 t}\right].$$

We can define the following coefficient: $D_n = \frac{1}{2}(a_n - jb_n)$, which we can write in terms of the integral formulas for a_n and b_n:

$$D_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[f(t)\cos(n\omega_0 t) - jf(t)\sin(n\omega_0 t)\right] dt.$$

Observe that using negative values of n will collapse into the other coefficient, the complex conjugate of the first. If $n = 0$, $D_0 = \frac{1}{T}\int_T f(t)\,dt = a_0$. This results in the form:

$$f(t) = D_0 + \sum_{n=1}^{\infty} \left[D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t}\right].$$

Definition 31. The exponential form of the Fourier Series is:

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}, n \in \mathbb{Z}.$$

For $D_n = \frac{1}{2}(a_n - jb_n), n \in \mathbb{Z}$ capturing the sine and cosine terms.

$$D_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-jn\omega_0 t}\, dt \quad \forall n \in \mathbb{Z}.$$

2026-02-11 **Lecture 17**

Consider a real function $f(t)$ with an exponential Fourier series. We can rewrite our expression for D_n, which uses coefficients from our Trigonometric Fourier Series, in terms of information from our compact Fourier Series:

$$D_n = \frac{c_n}{2}e^{j\phi_n}, \quad D_{-n} = \frac{c_n}{2}e^{-j\phi_n}, \quad a_0 = c_0 = D_0.$$

We can create a magnitude spectrum for the exponential form, plotting $|D_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} = \frac{c_n}{2}$. Note that $|D_n| = |D_{-n}|$ as they are each other's complex conjugate. The magnitude spectrum is therefore an even function of ω or n.

We can create a phase spectrum for the exponential form. Since $D_n = \frac{c_n}{2}e^{j\phi_n}$, and $D_{-n} = \frac{c_n}{2}e^{-j\phi_n}$, the phase spectrum is an odd function of ω or n.

2026-02-23 **Lecture 18**

Theorem 11. Parseval's theorem can be transformed to the exponential form of the Fourier Series:

$$P_f = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |D_n|^2.$$

2026-02-25 **Lecture 19**

5 The Fourier Transform

The Fourier series is applicable when signals are periodic, and absolutely integrable. We can solve both of these limitations using the Fourier and Laplace transforms.

Take a periodic and absolutely integrable function $f(t) = f(t+T)$, $\int_T |f(t)| \, dt < \infty$. We can represent this function with the exponential Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}, \quad D_n = \frac{1}{T}\int_T f(t)\, e^{-jn\omega_0 t} \, dt.$$

Let's redefine our discrete harmonics as a continuous frequency $n\omega_0 = \omega$. Note that $\Delta\omega = \omega_0$ Rewriting our form of the exponential Fourier series in terms of these new quantities:

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_T f(t)\, e^{-j\omega t} \, dt \cdot e^{j\omega t}.$$

Now, to remove the periodicity, we can take $T \to \infty$:

$$f(t) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} \, dt\right] \cdot e^{j\omega t} d\omega.$$

We can split this bulky equation into two parts.

Definition 32. The Fourier transform of $f(t)$ is defined as:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt.$$

The inverse Fourier transform of $F(\omega)$, or Fourier integral, is:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, dw.$$

We have turned $\omega \in \mathbb{R}$ into a continuous basis variable, which will give us a frequency representation of our original $f(t)$.

We have 'eliminated' the periodicity requirement of our function $f(t)$ by taking $T \to \infty$. However, we maintain that $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$. Additionally, $F(\omega)$ is in general complex:

$$F(\omega) = |F(\omega) e^{j\angle F(\omega)}| = \mathbb{R}e[F(\omega)] + j\mathbb{I}m[F(\omega)].$$

If we assume that $f(t)$ is real, then our Fourier transform in rectangular form is:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cos(\omega t)\, dt - j \int_{-\infty}^{\infty} f(t) \sin(\omega t)\, dt.$$

This implies that the real part is an even function, and the odd part is an odd function. Therefore, the magnitude will be an even function, while the phase will be an odd function.

Example. The Fourier transform of the delta function is:

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt = 1.$$

The inverse Fourier transform of the delta function in the frequency domain is:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t}\, dw = \frac{1}{2\pi}.$$

Intuitively, the delta function in the time domain is an infinitely thin function, requiring an infinitely wide frequency domain (similar to uncertainty principle $\Delta x \Delta k \geq C$). ♢

Example. Note that $\mathcal{F}\{\delta(t)\} = 1$ and $\mathcal{F}\{1\} = 2\pi\delta(\omega)$. Observe that the time-domain function 1 is not absolutely integrable, as it diverges. As such, we end up with a distribution as our Fourier transform. ♢

Lecture 20

2026-02-27

Example. Find the inverse Fourier transform of $\delta(\omega - \omega_0)$:

$$\mathcal{F}^{-1}\{\delta(\omega - \omega_0)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t}\, d\omega = \frac{1}{2\pi} e^{j\omega_0 t}.$$

Therefore, going backwards:

$$\mathcal{F}\left\{e^{-j\omega_0 t}\right\} = 2\pi\delta(\omega + \omega_0).$$

♢

From the above example, we can use our complex exponential formulas for sinusoids to obtain the results:

$$\mathcal{F}\left\{\cos \omega_0 t\right\} = \pi \left[\delta\left(\omega+\omega_0\right)+\delta\left(\omega-\omega_0\right)\right]$$
$$\mathcal{F}\left\{\sin \omega_0 t\right\} = j\pi \left[\delta\left(\omega+\omega_0\right)-\delta\left(\omega-\omega_0\right)\right].$$

Example. The Fourier transform of a constant $f\left(t\right)=k$:

$$\mathcal{F}\left\{k\right\} = 2\pi k\delta\left(\omega\right).$$

◊

Example. Find the Fourier transform of $f\left(t\right)=e^{-\alpha t}u\left(t\right), \alpha \in R^{+}$:

$$\mathcal{F}\left\{f\left(t\right)\right\} = \int_0^\infty e^{-\left(\alpha+j\omega\right)t}\,dt = \frac{1}{\alpha+j\omega}.$$

◊

From this result, we can find the magnitude and phase of our Fourier transform:

$$\left|F\left(\omega\right)\right| = \frac{1}{\sqrt{\alpha^2+\omega^2}}, \qquad \angle F\left(\omega\right) = \angle 1 - \angle \alpha + j\omega = -\arctan\frac{\omega}{\alpha}.$$

Noting that they produce and even and an odd function, respectively.

Example. Find the Fourier transform of the rectangular pulse.

$$F\left(\omega\right) = \mathcal{F}\left\{P_d\left(t\right)\right\} = \int_{-\frac{d}{2}}^{\frac{d}{2}} 1 \cdot e^{-j\omega t}\,dt = \frac{-1}{j\omega}\left[e^{-\frac{j\omega d}{2}} - e^{\frac{j\omega d}{2}}\right] = \frac{2}{\omega}\sin\frac{\omega d}{2}.$$

This can be rewritten as:

$$F\left(\omega\right) = d\operatorname{sinc}\frac{\omega d}{2}.$$

◊

2026-03-02

Lecture 21

Example. Find the Fourier series of the unit step function.

Note that this does not obey the Dirichlet conditions, since it isn't absolutely integrable. Using the direct approach, we get:

$$\mathcal{F}\left\{u\left(t\right)\right\} = -\frac{1}{j\omega}e^{-j\omega t}\Big|_0^\infty.$$

Which has no limit.

Instead, consider the approximation: $u\left(t\right) \approx \lim_{a\to 0^+} e^{-at}u\left(t\right)$. Now, evaluating our Fourier transform and taking the limit, we get:

$$\mathcal{F}\left\{u\left(t\right)\right\} = \pi\delta\left(\omega\right) + \frac{1}{j\omega}.$$

◊

5.1 Properties of the Fourier Transform

Linearity

The Fourier transform is a linear operator:

$$\mathcal{F}\left\{a_1 f_1(t) + a_2 f_2(t)\right\} = a_1 \mathcal{F}\left\{f_1(t)\right\} + a_2 \mathcal{F}\left\{f_2(t)\right\}.$$

This property follows from the linearity of integrals.

Scaling

The Fourier transform propagates scaling, in that a compression in the time domain results in an expansion in the frequency domain:

$$\mathcal{F}\left\{f(at)\right\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right), \quad F(\omega) = \mathcal{F}\left\{f(t)\right\}.$$

Proof Assume $a > 0, a \in \mathbb{R}^+$. Let $x = at, dx = adt$, $t \to \pm\infty \implies x \to \pm\infty$. Taking the Fourier transform:

$$\mathcal{F}\left\{f(at)\right\} = \int_{-\infty}^{\infty} f(at)\, e^{-j\omega t}\, dt = \frac{1}{a}\int_{-\infty}^{\infty} f(x)\, e^{-j\frac{\omega}{a}x}\, dx = \frac{1}{a} F\left(\frac{\omega}{a}\right).$$

If $a < 0$, then our limits flip but an additional negative pops out such that we get a coefficient $-\frac{1}{a}$ which is now positive. Therefore, we take the magnitude as our general case.

Time Reversal

$$\mathcal{F}\left\{f(-t)\right\} = F(-\omega).$$

Proof

Use $a = -1$ in the previous property.

First Translation Property: Time Shift

$$\mathcal{F}\left\{f(t - t_0)\right\} = F(\omega)\, e^{-j\omega t_0}.$$

Proof

$$\mathcal{F}\left\{f(t - t_0)\right\} = \int_{-\infty}^{\infty} f(t - t_0)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} f(x)\, e^{-j\omega x} e^{-j\omega t_0}\, dx = F(\omega)\, e^{-j\omega t_0}.$$

Note that this corresponds with a magnitude scale of unity. However, the phase of the Fourier transform corresponds with:

$$\angle \mathcal{F}\left\{f(t - t_0)\right\} = \angle F(\omega) - \omega t_0.$$

This effect can be a destabilizing effect of systems.

Example. Examine the Fourier transform of the rectangular pulse of width d centered at $t = 0$:

$$F(\omega) = d\operatorname{sinc}\left(\frac{\omega d}{2}\right).$$

If we shift the pulse $f\left(t - \frac{d}{2}\right)$ so it's left edge lies at $x = 0$:

$$F(\omega)\, e^{-j\omega\frac{d}{2}} = d\operatorname{sinc}\left(\frac{\omega d}{2}\right) e^{-j\omega\frac{d}{2}}.$$

This leaves the magnitude unchanged, but the phase shifted by $-\omega\frac{d}{2}$. $\Diamond$

2026-03-04 # Lecture 22

Second Translational Property: Frequency Shift

$$\mathcal{F}\left\{f(t)\, e^{j\omega_0 t}\right\} = F(\omega - \omega_0).$$

Proof

Take the Fourier transform, and note that this shift amounts to a notational change.

$$\mathcal{F}\left\{f(t)\, e^{j\omega_0 t}\right\} = \int_{-\infty}^{\infty} f(t)\, e^{-j(\omega-\omega_0)t}\, dt = F(\omega - \omega_0).$$

Example. Find the Fourier Transform of $f(t)\cos\omega_0 t$.

Observe that $\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$. Therefore, we can use our frequency shifting property to obtain:

$$\mathcal{F}\{f(t)\} = \frac{1}{2}\left(F(\omega-\omega_0) + F(\omega+\omega_0)\right).$$

$\Diamond$

Convolution

$$\mathcal{F}\{f_1(t) * f_2(t)\} = F_1(\omega)\, F_2(\omega).$$

Proof

$$\mathcal{F}\{f_1 * f_2\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau\right] e^{-j\omega t}\, dt.$$

We can revise the order of integration.

$$= \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} f_2(t-\tau)\, e^{-j\omega t}\, dt\right] d\tau.$$

We can do a change of variables to show that the inner integral is just the Fourier transform of f_2. Then, the leftover factor of $e^{-j\omega\tau}$ acts to leave the outer integral as the Fourier transform of f_1.

$$\therefore \mathcal{F}\{f_1 * f_2\} = F_1(\omega)\, F_2(\omega).$$

Example. Find the inverse Fourier transform if $F(\omega) = \frac{1}{(1+j\omega)^2}$ using convolution.

Note that $F(\omega) = F_1(\omega)^2$, where each $F_1(\omega)$ is the Fourier transform of $f_1(t) = e^{-t}u(t)$. Therefore, using the convolution property, the IFT of F is the convolution of f_1 with itself.

$$\mathcal{F}^{-1}\{F(\omega)\} = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)\, e^{\tau-t}u(\tau-t)\, d\tau = u(t)\int_0^t e^{-t}\, d\tau = te^{-t}u(t).$$

$\Diamond$

Convolution in frequency domain

Using the symmetry property, we can obtain:

$$\mathcal{F}\{f_1(t)\, f_2(t)\} = \frac{1}{2\pi}F_1(\omega) * F_2(\omega).$$

Lecture 23

2026-03-06

Time Differentiation

Take $F(\omega) = \mathcal{F}\{f(t)\}$. If $f' = \frac{df}{dt}$ exists, then:

$$\mathcal{F}\{f'(t)\} = j\omega F(\omega).$$

Moreover,

$$\mathcal{F}\left\{f^{(n)}(t)\right\} = (j\omega)^n(\omega).$$

Proof

Note that:

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega.$$

It is clear that differentiating both sides with respect to time will result in one power of $j\omega$ dropping down in front of $F(\omega)$.

Symmetry

Let $\mathcal{F}\{f(t)\} = F(\omega)$. Using change of variables, we can show that directly swapping, ω and t gives:

$$\mathcal{F}^{-1}\{f(-\omega)\} = \frac{1}{2\pi}F(t).$$

5.2 Frequency Response of an LTI System

Recall that an LTI system can be characterized by its impulse response $H : \delta(t) \to h(t)$. H acting on some arbitrary function is equivalent to the convolution of $f(t)$ with the impulse response.

$$y(t) = H\{f(t)\} = f(t) * h(t).$$

Let us take the Fourier transform of both sides:

$$\begin{aligned} Y(\omega) = \mathcal{F}\{y(t)\} &= H(\omega)\, F(\omega) \\ &= |H(\omega)||F(\omega)|\,(\angle H(\omega) + \angle F(\omega)). \end{aligned}$$

Definition 33. The *frequency response* of an LTI system is $H(\omega)$, which is a system property expressed as a ratio:

$$H(\omega) = \frac{Y(\omega)}{F(\omega)}.$$

Example. Imagine a series RL circuit with $R = 2\Omega$ and $L = 1H$.

To find the output $i(t)$ frequency response to the input $v(t)$ of the voltage source. Our differential equation is:

$$v(t) = iR + L\frac{di}{dt}.$$

Taking the Fourier transform:

$$V(\omega) = I(\omega)\, R + jL\omega I(\omega) \implies V(\omega) = (R + j\omega L)\, I(\omega).$$

We can find our frequency response as:

$$H(\omega) = \frac{1}{R + j\omega L}.$$

We can find our impulse response using $H(\omega) = \mathcal{F}^{-1}\{H(\omega)\} = e^{-2t}u(t)$. Therefore, given $v(t) = 2e^{-t}u(t)$, we can find our current by taking the Fourier transform, using our transfer function. This gives $I(\omega)$, which we can decompose with partial fractions, and take the inverse Fourier transform to obtain our answer. ♢

Example. Given an impulse response $h(t)$, we claim that

$$y(t) = h(t) * \cos\omega_0 t = |H(\omega_0)| \cos(\omega_0 t + \angle H(\omega_0)).$$

We can prove this via the frequency domain.

$$\begin{aligned} Y(\omega) &= H(\omega) \cdot (\pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)) \\ &= \pi(H(\omega_0)\delta(\omega - \omega_0) + H(-\omega_0)\delta(\omega + \omega_0)). \end{aligned}$$

Note that:

$$\cos(\omega t + \theta) = \cos\omega t \cos\theta - \sin\omega t \sin\theta.$$

Therefore, we can use our results from above to obtain:

$$\begin{aligned} \mathcal{F}\{\cos(\omega_0 t + \theta)\} &= \pi\cos\theta[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] - j\pi\sin\theta[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)] \\ &= \pi\left[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)\right]. \end{aligned}$$

♢

2026-03-09

Lecture 24

5.3 Distortion Free System

Definition 34. A distortion free system is a system in which the output signal is a precise replica, perhaps scaled and shifted, of the input signal.

Theorem 12. An LTI system is distortion free provided that:

$$H(\omega) = ke^{-j\omega t_0}.$$

Proof

Note that:

$$Y(\omega) = H(\omega)F(\omega) = kke^{-j\omega t_0}F(\omega).$$

Taking the inverse transform:

$$y(t) = \mathcal{F}^{-1}\{Y(\omega)\} = kf(t - t_0).$$

Let's examine an ideal low-pass filter. In the frequency domain, this appears as a rectangular pulse across our cutoff frequencies $\omega \in (-\omega_c, \omega_c)$. The phase shift of this function is the line: $-\omega t_0$.

$$H(\omega) = \begin{cases} e^{-j\omega t_0} & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}.$$

However, this filter is not physically implementable.

$$\begin{aligned} h(t) = \mathcal{F}^{-1}\{H(\omega)\} &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(t-t_0)}\,d\omega \\ &= \frac{1}{2\pi j(t-t_0)}\left[e^{j\omega_c(t-t_0)} - e^{-j\omega_c(t-t_0)}\right] \\ &= \frac{1}{\pi(t-t_0)}\sin(\omega_c(t-t_0)) \\ &= \frac{\omega_c}{\pi}\operatorname{sinc}(\omega_c(t-t_0)). \end{aligned}$$

Observe that this is a *non-causal* filter, which is not physically realizable.

Definition 35. *Paley-Wiener Criterion for Physical Implementation*: For something to be physically possible, it must be causal.

$$h(t) = 0 \forall t < 0.$$

Causality is equivalent to the following:

$$\int_{-\infty}^{\infty} \frac{\log_{10}|H(\omega)|}{1+\omega^2}\,d\omega < \infty.$$

If $|H(\omega)| = 0$ over a finite interval of frequencies, then the integral will go to infinity and is therefore not possible.

Lecture 25

2026-03-11

Theorem 13. Planchacel's Theorem states that:

$$E_f = \int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega.$$

Proof

The inverse Fourier transform of F is given by:

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{j\omega t}\,d\omega.$$

Taking the magnitude of f :

$$|f(t)|^2 = f \cdot f^*.$$

Taking the energy of this:

$$\begin{aligned} E_f &= \int_{-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)\,e^{-j\omega t}\,d\omega\right]dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)\int_{-\infty}^{\infty} f(t)\,e^{-j\omega t}\,dt\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)\,F(\omega)\,d\omega. \end{aligned}$$

Where $|F(\omega)|^2$ is the spectral energy density.

Example. Consider $f(t) = P_d(t)$. We know that $F(\omega) = d\text{sinc}\left(\frac{\omega d}{2}\right)$. The spectral energy density is:

$$|F(\omega)|^2 = d^2\text{sinc}^2\left(\frac{\omega d}{2}\right).$$

The majority of the spectral energy is stored in low ω. $\lozenge$

Example. For $f(t) = e^{-t}u(t)$, calculate:

$$\Delta E = \frac{\text{energy in } |\omega| < 4}{\text{total energy}}.$$

The total energy is given by:

$$E_f = \int_0^\infty e^{-2t}\,dt = \frac{1}{2}.$$

Alternatively, $F(\omega) = \frac{1}{1+j\omega}$. Calculating the total spectral energy:

$$E_f = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+\omega^2}\,d\omega = \frac{1}{2}.$$

Now, calculating the energy stored on our frequency interval.

$$E_4 = \frac{1}{2\pi}\int_{-4}^{4} \frac{1}{1+\omega^2}\,d\omega = \frac{1}{2\pi}(\arctan 4 - \arctan -4) = \frac{1}{\pi}\arctan 4.$$

$\lozenge$

Example. With $f(t) = e^{-t}u(t)$, find ω_0 such that 95% of E_f is contained in $|\omega| < \omega_0$.

From above, the total energy is $\frac{1}{2}$.

$$\Delta E = 0.475 = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \frac{1}{1+\omega^2}\,d\omega = \frac{1}{\pi}\arctan \omega_0.$$

This gives $\omega_0 = 12.7\frac{rad}{s}$. $\lozenge$

Most signals have bandwidths that stretch to infinity, with diminishing energy at increasingly high frequencies. Plotting $|F(\omega)|^2$ yields a curve with a y-asymptote at 0.

Definition 36. The *essential bandwidth* is defined as the range of $|\omega| < \omega_0$ that contains some net fraction of the total energy.

2026-03-13 **Lecture 26**

6 The Laplace Transform

To take the Fourier transform of a function, the function must be absolutely integrable. With systems, the impulse response being absolutely integrable implies that the system H must be BIBO stable. To address this limitation, we introduce the Laplace transform.

First, we must assume that our function is causal, $f(t) = 0 \forall t < 0$. The idea is that $f(t)e^{-s\infty} = 0$ for existence.

Definition 37. The Laplace transform is:

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt, \quad s = \sigma + j\omega.$$

The inverse Laplace transform is:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)\, e^{st}\, ds.$$

To take the Laplace transform, we must select σ to satisfy this condition:

$$\int_{-\infty}^{\infty} |f(t)\, e^{-\sigma t}|\, dt < \infty.$$

Example. Take $f(t) = e^{-at} u(t)$, find the Laplace transform and region of convergence for $a > 0$ and $a < 0$.

For $a > 0$, we observe that:

$$\int_{-\infty}^{\infty} |e^{-(a+\sigma)t}|\, dt < \infty \iff \sigma > -a.$$

The Laplace transform becomes:

$$F(s) = \int_{0^-}^{\infty} e^{-(a+s)t}\, dt = \frac{1}{a+s}.$$

For $a < 0$, we obtain the same condition:

$$\int_{-\infty}^{\infty} |e^{(|a|-\sigma)t}|\, dt < \infty \iff \sigma > |a| = -a.$$

The Laplace transform is otherwise the same. ◊

Lecture 27

2026-03-18

Let's find the Laplace transform of various signals.

- $\mathcal{L}\{\delta(t)\}$

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)\, e^{-st}\, dt = 1 \forall s \in \mathbb{C}.$$

- $\mathcal{L}\{u(t)\}$

$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} e^{-st}\, dt = -\frac{1}{s}(0-1) = \frac{1}{s}, \quad \sigma > 0 \text{ or } Re\{s\} > 0.$$

We can compare the Fourier transform and the Laplace transform *only* when we have a causal signal, otherwise the Laplace transform is not defined! The Laplace transform is given by:

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0}^{\infty} f(t)\, e^{-st}\, dt.$$

The Fourier transform, using the causality of the signal to simply the calculation, is:

$$\mathcal{F}\left\{f\left(t\right)\right\} = F\left(\omega\right) = \int_{0}^{\infty} f\left(t\right) e^{-j\omega t}\, dt.$$

This gives that for a causal signal, the Laplace and Fourier transforms are identical if we take $s = j\omega$ and the imaginary axis ($\sigma = 0$) is included in the region of convergence of the Laplace transform.

Example. Take $f\left(t\right) = u\left(t\right)$.

$$\mathcal{L}\left\{f\left(t\right)\right\} = \frac{1}{s}, \quad Re\left\{s\right\} = 0.$$

$$\mathcal{F}\left\{f\left(t\right)\right\} = \pi\delta\left(\omega\right) + \frac{1}{j\omega}.$$

Observe that the two results are clearly not the same, even through $u\left(t\right)$ is causal. The reason for this is that $\sigma > 0$ for the convergence of the Laplace transform, which does not include the imaginary axis! ◊

- $\mathcal{L}\left\{e^{\lambda t}u\left(t\right)\right\}$

$$\mathcal{L}\left\{e^{\lambda t}u\left(t\right)\right\} = \int_{0^-}^{\infty} e^{\lambda t}u\left(t\right)e^{-st}\, dt = \int_{0}^{\infty} e^{-(s-\lambda)t}\, dt.$$
$$= -\frac{1}{s-\lambda}e^{-(s-\lambda)t}|_0^\infty = \frac{1}{s-\lambda}, \quad Re\left\{s\right\} > Re\left\{\lambda\right\}.$$

- $\mathcal{L}\left\{tu\left(t\right)\right\}$

$$\mathcal{L}\left\{tu\left(t\right)\right\} = \int_{0}^{\infty} te^{-st}\, dt = \frac{t}{-s}e^{-st}|_0^\infty - \frac{1}{s^2}e^{-st}|_0^\infty = \frac{1}{s^2}, \quad Re\left\{s\right\} > 0.$$

- $\mathcal{L}\left\{u\left(t\right)\cos\left(\omega_0 t\right)\right\}$

$$\begin{aligned}
\mathcal{L}\left\{\cos\left(\omega_0 t\right)u\left(t\right)\right\} &= \frac{1}{2}\int_{0}^{\infty} e^{j\omega_0 t}e^{-st}\, dt + \frac{1}{2}\int_{0}^{\infty} e^{-j\omega_0 t}e^{-st}\, dt \\
&= -\frac{1}{2}\left[\frac{e^{-(s-j\omega)t}}{s-j\omega} + \frac{e^{-(s+j\omega)t}}{s+j\omega}\right]|_0^\infty \\
&= \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] \\
&= \frac{s}{s^2+\omega_0^2}, \quad Re\left\{s\right\} > 0.
\end{aligned}$$

- $\mathcal{L}\left\{u\left(t\right)\sin\left(\omega_0 t\right)\right\}$

$$\mathcal{L}\left\{u\left(t\right)\sin\left(\omega_0 t\right)\right\} = \frac{\omega_0}{s^2+\omega_0^2}.$$

6.1 Properties of the Laplace Transform

Linearity

$$\mathcal{L}\left\{af_1\left(t\right) + bf_2\left(t\right)\right\} = a\mathcal{L}\left\{f_1\left(t\right)\right\} + b\mathcal{L}\left\{f_2\left(t\right)\right\}.$$

Where the region of convergence is the intersection of the ROC of each function.

Lecture 28

2026-03-20

Time Shifting

$$\mathcal{L}\left\{f\left(t-t_0\right)\right\} = F\left(s\right)e^{-st_0}, \quad t_0 > 0.$$

Note that this means we must have $t_0 > 0$ to obtain a causal signal.

Frequency Shifting

$$\mathcal{L}\left\{f\left(t\right)\right\} = F\left(s\right) \implies \mathcal{L}\left(f\left(t\right)e^{\alpha t}\right) = F\left(s-\alpha\right).$$

Time Scaling

$$\mathcal{L}\left\{f\left(at\right)\right\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0.$$

Time Differentiation

$$\mathcal{L}\left\{f\left(t\right)\right\} = F\left(s\right) \implies \mathcal{L}\left\{\frac{df}{dt}\right\} = sF\left(s\right) - f\left(0^-\right).$$

Which we can prove via integration by parts. This represents the effect of an initial condition on f, particularly relevant for solving differential equations.

For higher order derivatives we more generally have:

$$\mathcal{L}\left\{f^{(n)}\left(t\right)\right\} = s^nF\left(s\right) - s^{n-1}f\left(0^-\right) - \ldots - sf^{(n-2)}\left(0^-\right) - f^{(n-1)\left(0^-\right)}.$$

Time Integration

$$\mathcal{L}\left\{\int_{0^-}^{\infty} f\left(x\right)dx\right\} = \frac{F\left(s\right)}{s}.$$

Proof Let $g\left(t\right) = \int_{0^-}^{\infty} f\left(x\right)dx$. We know that $g\left(0^-\right) = 0$, $f = \frac{dg}{dt}$:

$$\mathcal{L}\left\{\frac{dg}{dt}\right\} = sG\left(s\right) - g\left(0^-\right) = F\left(s\right) \implies G\left(s\right) = \frac{F\left(s\right)}{s}.$$

Frequency Differentiation

$$\mathcal{L}\left\{-tf\left(t\right)\right\} = \frac{d}{ds}F\left(s\right).$$

For for higher order differentiation:

$$\mathcal{L}\left\{\left(-t\right)^n f\left(t\right)\right\} = \frac{d^n}{ds^n}F\left(s\right).$$

Proof

$$F\left(s\right) = \int_{0^-}^{\infty} f\left(x\right)e^{-st}\,dt.$$

Taking the derivative with respect to s:

$$\frac{d}{ds}F\left(s\right) = \int_{0^-}^{\infty} -tf\left(t\right)e^{-st}\,dt.$$

Time Convolution

$$\mathcal{L}\left\{f_1(t) * f_2(t)\right\} = F_1(s)\, F_2(s)\,.$$

Frequency Convolution

$$\mathcal{L}\left\{f_1(t)\, f_2(t)\right\} = \frac{1}{2\pi j} F_1(s) * F_2(s)\,.$$

Note that this is an application of the symmetry property.

2026-03-23

Lecture 29

Example. For $f(t) = u(t) - u(t - T)$, find $g(t) = f(t) * f(t)$.

Note that $\mathcal{L}\left\{f(t)\right\} = \frac{1}{s}\left(1 - e^{-sT}\right) = F(s)$. In the frequency domain, a time domain convolution is simply a product. Therefore:

$$G(s) = \frac{1}{s^2} - 2\frac{e^{-sT}}{s^2} + \frac{e^{-2sT}}{s^2}.$$

Taking the inverse transform, we get:

$$g(t) = tu(t) - 2(t - T)\, u(t - T) + (t - 2T)\, u(t - 2T)\,.$$

◊

LTI Systems

Take an LTI system defined by it impulse response $h(t)$. Taking the Laplace transform of our system:

$$y(t) = h(t) * f(t) \to Y(s) = H(s)\, F(s)\,.$$

If $f(t) = \delta(t)$, then we trivially get $Y(s) = H(s)$. Recall the we define $H(s)$ as the *system transfer function*, or as the ratio of the frequency domain output to the input:

$$H(s) = \frac{Y(s)}{F(s)}.$$

The transfer function is a system property.

6.2 LTI ODEs

Systems defined by linear, time-invariant ordinary differential equations, using causal signals, can be solved using Laplace transforms. Note that since our input signals must be causal, $f(0^-) = 0$ always. When taking the Laplace transform, we will obtain an equation of the form:

$$Y(s) = \frac{N(s)}{Q(s)} + \frac{P(s)}{Q(s)} F(s)\,.$$

$Q(s)$ is defined by the coefficients of the differential equation, as it is that characteristic polynomial. $N(s)$ is defined by the initial conditions, which appear when taking the Laplace transform of a derivative. $P(s)$ is defined by the right hand side, or input side, of the ODE.

Note. The solution to our ODE in the frequency domain is composed of two terms: the first is the natural response, while the other is the forced response. △

Take a generalized LTI system described by the ODE:

$$\frac{d^n y}{d^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0 y = b_m \frac{d^m f}{dt^m} + b_{m-1}\frac{d^{m-1}f}{dt^{m-1}} + \ldots + b_0 f, \quad n \geq m.$$

Where the order of differentiation on the output must be greater than the order on the input. Taking the Laplace transform:

$$Q(s)\,Y(s) - N(s) = P(s)\,F(s).$$

We return to our general form earlier, with Q as the characteristic polynomial:

$$Q(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0.$$

P as the polynomial defined by the input:

$$P(s) = b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0.$$

And N is defined by the initial conditions of the problem, with order strictly less than Q:

$$N(s) = c_{n-1}s^{n-1} + \ldots + c_0.$$

Lecture 30

2026-03-25

We can take the inverse Laplace transform of each part of this solution. The *zero-input* response is given by:

$$y_0(t) = \mathcal{L}^{-1}\left\{\frac{N(s)}{Q(s)}\right\}.$$

And the *zero-state* response is given by:

$$y_f(t) = \mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}F(s)\right\}.$$

The total response $y_T(t) = y_0(t) + y_f(t)$ is the sum of each part.

Zero-state response

Imagine a system described by:

$$\frac{d^n y}{d^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0 y = b_m \frac{d^m f}{dt^m} + b_{m-1}\frac{d^{m-1}f}{dt^{m-1}} + \ldots + b_0 f, \quad n \geq m.$$

Let's examine the zero-state response, with the assumption that all initial conditions are 0. Taking the Laplace transform of our general differential equation.

$$\begin{gathered}\left[s^n + a_{n-1}s^{n-1} + \ldots + a_0\right]Y(s) = \left[b_n s^n + b_{n-1}s^{n-1} + \ldots + b_0\right]F(s)\\ Q(s)\,Y(s) = P(s)\,F(s).\end{gathered}$$

Our system transfer function:

$$H(s) = \frac{P(s)}{Q(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}.$$

Depends only on the system's ODE. We can obtain the impulse response as:

$$h(t) = \mathcal{L}^{-1}\left\{H(s)\right\}.$$

Example. Given the system transfer function $H(s)$, find the impulse response $h(t)$, the system's ODE, the zero state response for $f(t) = e^{-t}u(t)$, and the total response for $y(0^-) = 1$, $\dot{y}(0^-) = 1$.

$$H(s) = \frac{1}{s^2 + 5s + 6}.$$

First, the impulse response can be found as:

$$h(t) = \mathcal{L}^{-1}\{H(s)\}, \quad H(s) = \frac{1}{s+2} - \frac{1}{s+3}.$$

Taking an inverse transform:

$$h(t) = \left[e^{-2t} - e^{-3t}\right] u(t).$$

Note that we found earlier:

$$H(s) = \frac{Y(s)}{F(s)} = \frac{P(s)}{Q(s)} \implies \left(s^2 + 5s + 6\right) Y(s) = F(s).$$

Taking the inverse transform:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = f.$$

Therefore, given an input $f(t) = e^{-t}u(t)$ and taking the Laplace transform $F(s) = \frac{1}{s+1}$ yields:

$$Y(s) = \frac{1}{(s+2)(s+3)} \cdot \frac{1}{s+1}.$$

Taking partial fractions and then an inverse Laplace transform:

$$y_0(t) = \left[\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} - e^{-2t}\right] u(t).$$

Now adding in our initial conditions, we can find the total response by taking the Laplace transform of our differential equation, *considering the presence of initial conditions*, and subbing in values:

$$\left[s^2 Y(s) - sy\left(0^-\right) - y'\left(0^-\right)\right] + 5\left[sY(s) - y\left(0^-\right)\right] + 6Y(s) = F(s).$$

$$\left(s^2 + 5s + 6\right) Y(s) - s - 6 = F(s).$$

Solving for our output, we obtain it as the inverse Laplce transform of two terms: the first being the zero-state response which we solved earlier, and the second being the zero-input response due to initial conditions:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{F(s)}{s^2 + 5s + 6}\right\} + \mathcal{L}^{-1}\left\{\frac{s+6}{s^2 + 5s + 6}\right\}.$$

Computing this yields the net response. ◊

BIBO Stability

We know that a system H is BIBO stable $\iff$ $h(t)$ is absolutely bounded:

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty.$$

Theorem 14. An LTI system with transfer function $H(s)$ given by:

$$H(s) = \frac{P(s)}{Q(s)} = \frac{\beta_n s^n + \ldots + \beta_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}.$$

Is BIBO stable $\iff$ the roots of $Q(s)$ are strictly in the left half of the s-plane (i.e. $Re\{\text{roots}\} < 0$).

Lecture 31

2026-03-27

Proof of Stability

Take $H(s) = P(s)/Q(s)$. Taking the inverse Laplace transform of the system transfer function:

$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\beta_n + \frac{b_{n-1}s^{n-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}\right\}.$$

Where β_n appears via division of polynomials.

1. If the roots of Q are real, simple (multiplicity of 1), and negative, then applying partial fractions and taking the inverse transform will result in decaying exponentials;
2. If the roots of Q have a multiplicity of greater than 1, but the real parts are negative, the exponential will decay quicker than the polynomial and the solutions will remain bounded;
3. If the roots come in complex conjugate pairs, we will obtain damped exponentials, where again if the roots have a negative real part, the solutions will be bounded.

Note. If the roots of $Q(s)$ have a negative real part, we can obtain the frequency response (Fourier transform of $h(t)$)$H(\omega)$ using: $H(s)|_{s=j\omega}$ △

Proof

Take system H as BIBO stable, meaning it has a Fourier transform. The frequency response of the system is given by:

$$\mathcal{F}\{h(t)\} = H(\omega).$$

This means $j\omega \in$ ROC.

6.3 Inverse Laplace Transform

Consider $\overline{F}(s) = F(s)e^{\alpha s}$. To find the inverse transform, first we examine the inverse transform of $F(s)$, then we apply the delay. Note that $F(s)$ must be strictly proper, meaning the degree of the denominator is higher than the degree of the numerator.

At this point, we can apply partial fractions.

Note. A cool trick when applying partial fractions is multiplying by one of the denominator terms, and subbing in it's root as a value of s. Although this may not be new, what is clever is that for higher multiplicity, we could then differentiate with respect to s to select out more individual coefficients with lower powers in the denominator. See this case where λ has multiplicity of 2:

$$F(s) = \frac{a_0}{(s-\lambda)^2} + \frac{a_1}{(s-\lambda)}.$$

Multiplying by $(s-\lambda)^2$ will select a_0, and then differentiating with respect to s will kill a_0 and select a_1. △

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Take $F(s) = \frac{P(s)}{Q(s)}$. If Q has complex conjugate roots:

$$Q(s) = (s + a + jb)(s + a - jb) = (s + a)^2 + b^2.$$

Then we note that our time-domain function will be the product of an exponential and a sinusoid. We can then deconstruct our F using partial fractions as a linear combination of our sin and our cos terms:

$$F(s) = k_1 \frac{s + a}{(s + a)^2 + b^2} + k_2 \frac{b}{(s + a)^2 + b^2}.$$

Where the first term transforms to a cosine. We can extract k_1 using a clever limit:

$$\lim_{s \to \infty} sF(s) = k_1.$$

And:

$$\lim_{s \to 0} F(s) = ak_1 + bk_2.$$

2026-04-01 **Lecture 33**

7 Low Pass Filter Design

The ideal filter has unity gain for pass-band frequencies, and complete attenuation beyond the cut-off frequency ω_c. In reality, our strict cut-off frequency is actually a gradual roll-off. We denote ω_p as the end of the pass-band, and ω_s as the start of the stop-band. For $\omega_p < \omega < \omega_s$, we have the transition band.

Note that we also include some bounds on the allowed magnitude deviation in each of our bands. While the ideal pass-band has $|\tilde{H}(j\omega)| = 1$, we allow sagging down to $|\tilde{H}(j\omega)| > \frac{1}{1+\epsilon^2}$. Additionally, in the stop-band we ideally want a transfer function magnitude of 0. However, we allow magnitudes up to $|\tilde{H}(j\omega)| < \frac{1}{\delta^2}$

We take the general Butterworth approximation to be:

$$|\tilde{H}_N(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2N}}.$$

This approaches our ideal 'step-down' filter model as N becomes very large. At $\omega = 0$, $|\tilde{H}(j\omega)| = 1$. At $\omega = \omega_c$, $|\tilde{H}(j\omega)| = \frac{1}{2}$. Also note that $|\tilde{H}(j\omega)|$ is monotonically decreasing, bounded above and below by 1 and 0, respectively.

Assuming that the filter is BIBO stable, we can substitute $s = j\omega$ into our filter. Also note that:

$$|\tilde{H}_N(j\omega)|^2 = \tilde{H}_N(j\omega)\,\tilde{H}_N(-j\omega).$$

Therefore, making our s substitution:

$$|\tilde{H}_N(s)|^2 = \frac{1}{1 + \left(\frac{s}{j\omega_c}\right)^{2N}}.$$

The roots of the denominator are given by:

$$s^{2N} = (-1)\,(j\omega_c)^{2N} = e^{-j\pi} e^{j2\pi k} e^{j\pi N} \omega_c^{2N} = e^{j\pi(2k+N-1)} \omega_c^{2N}$$
$$s_k = e^{j\pi\left(\frac{2k+N-1}{2N}\right)} \omega_c.$$

The magnitude of each of these roots is $|s_k| = \omega_c$, and the phase is $\angle s_k = \pi\left(\frac{2k+N-1}{2N}\right)$. This implies that all of the $2N$ roots are distributed across a circle in the complex plane, at a distance of ω_c from the origin. Their phase angles are evenly distributed around the complex circle of radius ω_c. Even N never has poles along the real axis, while odd N do.

To make a stable filter, we must isolate our poles on the left-hand side of the complex plane. For convenience, we normalize our filter to $\omega_c = 1$. Re-examining the poles of our filter:

$$s_k = \exp\left\{ j\pi\left(\frac{2k+N-1}{2N}\right) \right\}.$$

And taking only the poles on the left-hand side of the plane, we obtain:

$$= \frac{1}{(s-s_1)\,(s-s_2)\ldots(s-s_N)}.$$

Example. Find the normalized $N = 2$ Butterworth filter.

$$s_k = \exp\left[j\pi\left(\frac{2K+1}{4}\right) \right].$$

With $\omega_c \equiv 1$. This filter has poles at $\angle s_k = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ Therefore, we take our filter to be:

$$\tilde{H}_{\text{normalized}} = \frac{1}{\left(s - e^{j\frac{3\pi}{4}}\right)\left(s - e^{j\frac{5\pi}{4}}\right)}.$$

Expanding, we get:

$$\tilde{H}_{\text{normalized}}\,(s) = \frac{1}{s^2 + \sqrt{2}s + 1}.$$

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Lecture 34

2026-04-08

How do we de-normalize a filter? We can apply a horizontal scale by a factor of ω_c.

Example. Find a low-pass Butterworth filter with $N = 2$ and $\omega_c = 2\frac{rad}{s}$.

We have the normalized filter:

$$\tilde{H}_{\text{normalized}}\,(s) = \frac{1}{s^2 + \sqrt{2}s + 1}.$$

Recall that we obtained this result using the poles $s^{2N} = 0\,(j\omega_c)^{2N}$. Substitute s with $\frac{s}{\omega_c}$ in our normlized filter to shift ω_c. Therefore, the filter we seek is:

$$\tilde{H}\,(s) = \frac{4}{s^2 + 2\sqrt{2}s + 4}.$$

◊

Let's use our design considerations to design a realistic filter. In the pass-band, we want $|\tilde{H}(j\omega)|^2 > \frac{1}{1+\epsilon^2}$. Therefore, on the pass-band boundary, $|\tilde{H}(j\omega_p)|^2 \geq \frac{1}{1+\epsilon^2}$. We introduce a quantity called the stop-band ripple coefficient:

$$R_p = -10\log_{10}\frac{1}{1+\epsilon^2}.$$

Defining the maximum decibel attenuation allowed in our pass-band.

$$R_p = -10\log_{10}\frac{1}{1+\epsilon^2} = -10\log_{10}\frac{1}{1+\left(\frac{\omega_p}{\omega_c}\right)^{2N}}.$$

Solving, we obtain:

$$\left(\frac{\omega_p}{\omega_c}\right)^{2N} = 10^{\frac{R_p}{10}} - 1.$$

For our stop-band restriction, we want frequencies in the stop-band to be attenuated greater than $\frac{1}{\delta^2}$. We introduce the stop-band ripple coefficient:

$$R_s = -10\log_{10}\frac{1}{\delta^2}.$$

Defining the minimum decibel attenuation allowed in our stop-band.

$$R_s = -10\log_{10}\frac{1}{\delta^2} = -10\log_{10}\frac{1}{1+\left(\frac{\omega_s}{\omega_c}\right)^{2N}}.$$

Solving, we similarly get:

$$\left(\frac{\omega_s}{\omega_c}\right)^{2N} = 10^{\frac{R_p}{10}} - 1.$$

If we divide our two constraint equations, we obtain:

$$\left(\frac{\omega_p}{\omega_s}\right)^{2N} = \frac{10^{R_p/10}-1}{10^{R_s/10}-1}.$$

We solve for N, finally finding:

$$N = \frac{\log_{10}\left[\left(10^{R_p/10}-1\right)\left(10^{R_s/10}-1\right)^{-1}\right]}{2\log_{10}\frac{\omega_p}{\omega_s}}.$$

In general, $N \in \mathbb{R}$, so we always round up.

To choose ω_c, we can use our pass-band and stop-band equations to back calculate, but they will give different values! We can choose any $\omega : \omega_c^1 < \omega_c < \omega_c^2$. Note that this may result in failing our desired specifications, so we may have to compensate with a higher N.